CALCULUS II, SUMMER 2015 - WEEKEND PROBLEM SET 3

`60 points total = 50 points + 10 extra credit points`

`Name:` ____ `Score:` ______/ 50

Use this worksheet as the cover sheet for your write-up: write your name on this page, and staple this sheet to the front of your homework packet.

Please indicate clearly which problems you have worked on. You will receive no credit for submitting solutions that the grader cannot read and understand—be sure to write legibly!

1. Radioactive Decay (10 points)

Let $y = f(t)$ denote the amount of radioactive particles present at time t. The *universal law of radioactive decay*[1] states that

$$y' = -ky, \tag{1.1}$$

where the *decay constant* $k > 0$ is determined by the type of particles in question.

Problem 1.1 (2 points)**.** What is the solution of (1.1), given the initial condition $y(0) = y_0$?

The *half-life* of a radioactive element is the unique number t_H such that

$$\frac{y(t_H)}{y(0)} = \frac{1}{2}.$$

Problem 1.2 (2 points)**.** What is t_H in terms of k?

Problem 1.3 (2 points)**.** Show that

$$\frac{y(t + t_H)}{y(t)} = \frac{y(t_H)}{y(0)}$$

for all t.

Problem 1.4 (2 points)**.** Fix a decay constant k. Let t_H be the half-life obtained from (1.1) with initial condition $y = y_0$. Similarly, let $\widetilde{t_H}$ be the half-life obtained from (1.1) with initial condition $y = y_1$. Show that $t_H = \widetilde{t_H}$.

Problem 1.5 (2 points)**.** The half-life for Polonium-210 is approximately 140 days. Find what percentage of a given quantity of Polonium-210 remains after 1 year. (*Hint*: Use a calculator.)

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[1] The law is derived from the assumption that every particle is equally likely to decay at any moment in time. Therefore, the rate of decay $-\frac{dy}{dt}$ must be proportional to y, the number of particles.

2. Free Fall (15 points)

Recall that the *displacement function* $s(t)$ of an object gives the distance, at time t, between the object and the starting point. The *velocity function* $v(t)$ is given by the formula

$$v(t) = s'(t),$$

and the *acceleration function* $a(t)$ is given by the formula

$$a(t) = s''(t).$$

Suppose that we drop an object of mass m in the Earth's atmosphere. The two major forces acting on the body are the gravitational force mg, where $g > 0$ is the gravitational acceleration constant determined by the location on Earth, and air resistance[2] kv, where the air resistance constant $k > 0$ is determined by various physical factors that we do not wish to discuss here. For the sake of simplicity, we ignore all other forces in our model.

The gravitational force pulls the object to the ground, and the air resistance pushes the object away from the earth, and so the net force exerted on the object is

$$F = mg - kv.$$

By Newton's second law $F = ma$,

$$ma = mg - kv.$$

Since $a = v'$, we obtain the simplified *free fall equation*

$$mv' = mg - kv. \tag{2.1}$$

Problem 2.1 (5 points)**.** Solve (2.1) with the initial value $v(0) = 0$, and compute the displacement function $s(t)$.

Problem 2.2 (5 points)**.** Solve (2.1) with the initial value $v(0) = v_0$, and compute the displacement function $s(t)$.

Problem 2.3 (2 points)**.** The *terminal velocity* of an object in free fall is defined to be the limit

$$v_T = \lim_{t\to\infty} v(t).$$

Compute the terminal velocity of an object in free fall with the initial velocity $v(0) = v_0$. Does the terminal velocity change depending on the value of v_0?

Problem 2.4 (3 points)**.** An object of mass 1000 kg is thrown downwards from a helicopter at altitude 5000m. If the initial velocity of the object is 10 m/s, what is the altitude of the object after 1 minute? For simplicity's sake, use $g = 10$ and $k = 1$. (*Hint*: use a calculator.)

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[2]The kv term comes from a particularly simple form of the *drag equation*, which describes the resistance applied to an object moving through fluid.

3. Simple Harmonic Motion (15 points)

Recall that the *displacement function* $s(t)$ of an object gives the distance, at time t, between the object and the starting point. The *velocity function* $v(t)$ is given by the formula

$$v(t) = s'(t),$$

and the *acceleration function* $a(t)$ is given by the formula

$$a(t) = s''(t).$$

Consider a spring hooked to the ceiling with an object of mass m attached to the bottom of it. *Hooke's law*[3]

$$F = -ks,$$

where the *spring constant* $k > 0$ depends on the type of spring, describes the restoring force of the spring; here we consider the displacement of the bottom end of the spring. By Newton's second law $F = ma$,

$$ma = -ks.$$

Since $a = s''$, we obtain the *motion equation for simple harmonic motion*

$$ms'' = -ks. \tag{3.1}$$

Problem 3.1 (5 points)**.** Set $\omega = \sqrt{\frac{k}{m}}$, so that (3.1) can be written as follows:

$$s'' = -\omega^2 s.$$

Show that the general solution of (3.1) is of the form

$$s = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

for arbitrary constants c_1 and c_2. Show that the above solution can be written as

$$s = A\cos(\omega t - \phi),$$

where

$$A = \sqrt{c_1^2 + c_2^2} \text{ and } \varphi = \arctan \frac{c_2}{c_1}.$$

We say that A is the *amplitude* of the motion, ω the *angular frequency* of the motion, and φ the *phase* of the motion.

A *simple harmonic oscillator* is any object in motion that satisfies the equation of motion (3.1). Another example of a simple harmonic oscillator is a simple pendulum, provided that the range of angle of oscillation is small[4].

Problem 3.2 (10 points)**.** Consider a simple harmonic oscillator. Suppose that its initial displacement is 1, its initial velocity is 2, and its initial acceleration is -12. Compute the displacement and acceleration of the simple harmonic oscillator when the velocity is $\sqrt{8}$.

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[3]Hooke's law is obtained by considering the first-order term of the Taylor approximation of the restoring force. By the physical assumptions, the zeroth term is zero, and the higher-order terms as suitably small.

[4]The equation of motion in the general case is $\theta'' + \frac{g}{l}\sin\theta = 0$, where g is the gravitational acceleration constant and l is the length of the pendulum. If we assume that $|\theta| \leq c \ll 1$, then $\sin\theta$ can be approximated by θ, and so we obtain $\theta'' + \frac{g}{l}\theta = 0$, the motion equation for simple harmonic motion.

4. Damped Harmonic Oscillator (20 points)

Oscillators are often subjected to various external forces. As in Section 2, we shall consider just one additional force: air resistance[5] $cv = cs'$. The adding of external force results in the following modified form of (3.1):

$$ms'' + cs' + ks = 0.$$

Letting $\omega = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2m\omega}$, we can rewrite the above equation to obtain the *motion equation for damped harmonic oscillation*:

$$s'' + 2\zeta\omega s' + \omega^2 s = 0. \tag{4.1}$$

Problem 4.1 (5 points)**.** ζ in (4.1) is called the *damping constant*.

(1) Solve (4.1) when $\zeta > 1$. This case is referred to as *overdamped* and models the situation where the system reaches an equilibrium without oscillating.
(2) Solve (4.1) when $\zeta = 1$. This case is referred to as *critically damped* and models the situation where the system reaches an equilibrium as rapidly as possible without oscillating.
(3) Solve (4.1) when $\zeta < 1$. This case is referred to as *underdamped* and models the situation where the system oscillates with gradually decreasing amplitude.

If the system is underdamped, then the general form of the displacement function is of the form

$$s = e^{at}\left(c_1 \sin(bt) + c_2 \cos(bt)\right)$$

for appropriately chosen constants a and b. The *maximum amplitude* of the oscillator is

$$A_{\max}(t) = \max(e^{at}c_1, e^{at}c_2),$$

which should decrease over time. The *damped angular frequency* of the oscillator is

$$\omega_1 = b.$$

Problem 4.2 (15 points)**.** A damped harmonic oscillator consists of an object of mass 100 attached to a spring whose spring constant is 10000. The initial displacement of the oscillator is 3 and the initial velocity is 0. Suppose that the maximum amplitude of the oscillator decreased to half the initial value after 10 seconds of oscillation. Compute the damping constant and the damped angular frequency of the oscillator.

5. Further Reading

A classic introductory textbook on ordinary differential equations is *Ordinary Differential Equations* by Morris Tenenbaum and Harry Pollard (1963, Dover). It helps to know some linear algebra, and Gilbert Strang's book, *Introduction to Linear Algebra* (2009, Wellesley–Cambridge) is an excellent introduction to the subject. It might also be worth checking out Strang's new book, *Differential Equations and Linear Algebra* (2014, Wellesley–Cambridge).

Many more examples of modeling natural phenomena with differential equations can be found in various physical sciences textbooks. A nice introductory textbook that should be accessible to you by the end of this course is *An Introduction to Mechanics* by Daniel Kleppner and Robert J. Kolenkow (2010, Cambridge).

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[5]While we have chosen to incorporate air resistance, the damped harmonic oscillator model can account for other types of external forces, such as friction.