CALCULUS II, SUMMER 2015 - SEQUENCES

1. Sequences

The reader is surely familiar with the concept of a finite list of numbers:

$$a_1, a_2, \ldots, a_N.$$

In mathematical contexts, however, we often encounter *infinite* lists of numbers, such as the list of all multiples of 2:

$$2, 4, 8, 10, \ldots, 2^n, \ldots$$

The notion of *sequence* formalizes the "infinite list of numbers" concept, without relying on ambiguous notations such as $\cdots$. Here we recall that $\mathbb{N}$ is the set of all *natural numbers*:

$$1, 2, 3, 4, \ldots.$$

Definition 1.1. A *sequence* is a function whose domain is $\mathbb{N}$.

There are a number of standard notations for the sequence

$$a_1, a_2, \ldots, a_n, \ldots.$$

We list several of them here:

$$(a_n)_{n=1}^{\infty}, \{a_n\}_{n=1}^{\infty}, (a_n)_{n\geq 1}, \{a_n\}_{n\geq 1}, (a_n)_{n\in\mathbb{N}}, \{a_n\}_{n\in\mathbb{N}}.$$

The next example illustrates how to manipulate the indexing variable n and the initial index 1.

Example 1.2. We define a sequence $(a_n)_{n=1}^{\infty}$ by setting $a_n = \frac{1}{n}$ for each $n \geq 1$. The first few terms of $(a_n)_{n=1}^{\infty}$ are

$$\underbrace{\frac{1}{1}}_{n=1}, \underbrace{\frac{1}{2}}_{n=2}, \underbrace{\frac{1}{3}}_{n=3}, \ldots.$$

Using k instead of n does not change the sequence: the first few terms of $(a_k)_{k=1}^{\infty}$ are

$$\underbrace{\frac{1}{1}}_{k=1}, \underbrace{\frac{1}{2}}_{k=2}, \underbrace{\frac{1}{3}}_{k=3}, \ldots.$$

On the other hand, $(a_j)_{j=2}^{\infty}$ starts counting the index at 2, and so the first few terms of it are

$$\underbrace{\frac{1}{2}}_{j=2}, \underbrace{\frac{1}{3}}_{j=3}, \underbrace{\frac{1}{4}}_{j=4}, \ldots.$$

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Although $(a_{i+1})_{i=1}^{\infty}$ starts counting the index at 1, the subscript $i+1$ forces the first few terms of the sequence to be

$$\underbrace{\frac{1}{2}}_{i=1}, \underbrace{\frac{1}{3}}_{i=2}, \underbrace{\frac{1}{4}}_{i=3}, \dots.$$

The subscript could, of course, be manipulated further, so long as the formula we use produces natural-number values in all cases. For example, the first few terms of $(a_{2^m})_{m=1}^{\infty}$ are

$$\underbrace{\frac{1}{2}}_{m=1}, \underbrace{\frac{1}{4}}_{m=2}, \underbrace{\frac{1}{8}}_{m=3}, \dots.$$

□

Note that we have obtained a number of new sequences from $(a_n)_{n=1}^{\infty}$ above, all of which are, in a sense, *contained* in $(a_n)_{n=1}^{\infty}$. We now formalize this notion.

Definition 1.3. We say that $(b_n)_{n=1}^{\infty}$ is a *subsequence* of $(a_n)_{n=1}^{\infty}$ in case there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

$$a_{f(n)} = b_n$$

for all $n \in \mathbb{N}$. If, in particular, f is given by the formula $f(n) = n$, then we say that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are *equal*. We write

$$(a_n)_{n=1}^{\infty} = (b_n)_{n=1}^{\infty}$$

to indicate that the two sequences are equal.

Example 1.4. Let $(a_n)_{n=1}^{\infty}$ be a sequence. $(a_{n+1})_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. Indeed, $f(n) = n+1$ is a strictly increasing function of natural numbers, and

$$(a_{f(n)})_{n=1}^{\infty} = (a_{n+1})_{n=1}^{\infty},$$

as was to be shown. □

Example 1.5. Let $(a_n)_{n=1}^{\infty}$ be a sequence. $(a_n)_{n=2}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. To see this, we first note that

$$(a_n)_{n=2}^{\infty} = (a_{n+1})_{n=1}^{\infty}.$$

Now, $(a_{n+1})_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$, as per Example 1.4. □

Example 1.6. Let $(a_n)_{n=1}^{\infty}$ be a sequence. $(a_{2^n-7})_{n=3}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. To see this, we first note that

$$(a_{2^n-7})_{n=3}^{\infty} = (a_{2^{n+2}-7})_{n=1}^{\infty}.$$

Now, the function $f(n) = 2^{n+2} - 7$ is strictly increasing. Moreover, $f(1) = 2^{1+2} - 7 = 1$, and so $f(n) \geq 1$ for all $n \in \mathbb{N}$. It follows that f is a strictly increasing function of natural numbers. Now,

$$(a_{f(n)})_{n=1}^{\infty} = (a_{2^n-7})_{n=3}^{\infty},$$

and so $(a_{2^n-7})_{n=3}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. □

Exercise 1.7. Determine whether $(b_n)_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$.

(1) $a_n = \frac{1}{n}$, $b_n = \frac{1}{n+3}$
(2) $a_n = \frac{(-1)^n}{n}$, $b_n = \frac{1}{3^n}$

(3) $a_n = 3^n \log_2 n$, $b_n = n9^n$
(4) $a_n = \frac{n}{n+1}$, $b_n = \frac{2^n}{n+1}$

Another way of obtaining a new sequence from old sequences is to apply arithmetic operations on them:

Definition 1.8. Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be sequences.

(a) We define the *sum* of $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ to be the sequence $(a_n + b_n)_{n=1}^\infty$.
(b) We define the *difference* of $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ to be the sequence $(a_n - b_n)_{n=1}^\infty$.
(c) We define the *product* of $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ to be the sequence $(a_n b_n)_{n=1}^\infty$.
(d) If each term of $(b_n)_{n=1}^\infty$ is nonzero, then we define the *quotient* of $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ to be the sequence $(\frac{a_n}{b_n})_{n=1}^\infty$.
(e) Given a number c, we define the *scalar multiplication* of c and $(a_n)_{n=1}^\infty$ to be the sequence $(ca_n)_{n=1}^\infty$.
(f) Given a positive number p, we define the *exponentiation* of $(a_n)_{n=1}^\infty$ by p to be the sequence $(a_n^p)_{n=1}^\infty$.

Exercise 1.9. If $a_n = \sin n$ and $b_n = \cos n$ for each $n \geq 1$, what is $(2a_n b_n)_{n=1}^\infty$? What is $(\frac{2a_n}{b_n})_{n=1}^\infty$? Which one of these two is a subsequence of $(a_n)_{n=1}^\infty$?

Exercise 1.10. Set $a_n = \ln \sqrt{en}$ for each $n \geq 1$. What is $(2a_n)_{n=1}^\infty$? If we set $b_n = n^2$ for each $n \geq 1$, then is $(b_n)_{n=1}^\infty$ a subsequence of $(a_n)_{n=1}^\infty$?

2. Interlude: the Triangle Inequalities

A large part of the study of sequences involves procuring various *estimates*—the meaning of which will be clearer in the subsequent sections. In this section, we study the most fundamental estimation tools in mathematics: the triangle inequalities.

In its most basic form, the triangle inequality states that

$$|a + b| \leq |a| + |b|$$

whenever a and b are real numbers. This inequality can be extended for multiple numbers:

Theorem 2.1 (Triangle inequality)**.** *If $a_1, \ldots, a_N$ are real numbers, then*

$$\left| \sum_{n=1}^{N} a_n \right| \leq \sum_{n=1}^{N} |a_n|.$$

To establish the triangle inequality, we need the following preliminary fact:

Lemma 2.2. *$x \leq |x|$ for all real numbers x.*

Proof of lemma. If $x \geq 0$, then $x = |x|$. If $x < 0$, then $x < -x = |x|$. □

Proof of the triangle inequality. We suppose for now that $N = 2$. Observe that

$$|a_1 + a_2|^2 = (a_1 + a_2)^2 = a_1^2 + 2a_1 a_2 + a_2^2 = |a_1|^2 + 2a_1 a_2 + |a_2|^2$$

Lemma 2.2 implies that

$$|a_1|^2 + 2a_1 a_2 + |a_2|^2 \leq |a_1|^2 + 2|a_1 a_2| + |a_2|^2 = |a_1|^2 + 2|a_1||a_2| + |a_2|^2 = \left(|a_1| + |a_2|\right)^2.$$

Therefore,

$$|a_1 + a_2|^2 \leq \left(|a_1| + |a_2|\right)^2,$$

and taking the square root of both sides yields

$$|a_1 + a_2| \leq |a_1| + |a_2|.$$

We now suppose that $N > 2$. We shall assume that

$$\left| \sum_{n=1}^{N-1} a_n \right| \leq \sum_{n=1}^{N-1} |a_n|$$

and show that

$$\left| \sum_{n=1}^{N} a_n \right| \leq \sum_{n=1}^{N} |a_n|.$$

Since we have already proved the triangle inequality for the $N = 2$ case, the above process shows that the triangle inequality holds for the $N = 3$ case. The same logic then establishes the $N = 4$ case, the $N = 5$ case, and so on. This line of reasoning is called *mathematical induction*.

Because the triangle inequality holds in the $N = 2$ case,

$$\left| \sum_{n=1}^{N} a_n \right| = \left| \left(\sum_{n=1}^{N-1} a_n \right) + a_N \right| \leq \left| \sum_{n=1}^{N-1} a_n \right| + |a_N|.$$

Since we have assumed that

$$\left| \sum_{n=1}^{N-1} a_n \right| \leq \sum_{n=1}^{N-1} |a_n|,$$

we now have that

$$\left| \sum_{n=1}^{N} a_n \right| \leq \left| \sum_{n=1}^{N-1} a_n \right| + |a_N| \leq \left(\sum_{n=1}^{N-1} |a_n| \right) + |a_N| + \sum_{n=1}^{N} |a_N|,$$

as was to be shown. □

The triangle inequality establishes an *upper bound* of the quantity $|a_1 + \cdots + a_N|$. A simple modification yields a lower bound as well:

Theorem 2.3 (Reverse triangle inequality)**.** *If a and b are real numbers, then*

$$|a - b| \geq \left| |a| - |b| \right|.$$

Proof. By the triangle inequality (Theorem 2.1),

$$|a| = |(a - b) + b| \leq |a - b| + |b|.$$

Therefore,

$$|a| - |b| \leq |a - b|.$$

Similarly, the triangle inequality implies that

$$|b| = |(b - a) + a| = |b - a| + |a| = |a - b| + |a|,$$

and so

$$|b| - |a| \leq |a - b|.$$

It now follows that

$$||a| - |b|| \leq |a - b|,$$

as was to be shown. □

With these tools at hand, we can now return to the study of sequences.

3. Limit of a Sequence

We say that a sequence $(a_n)_{n=1}^\infty$ is *eventually constant* if there exist a real number L and a positive integer N such that

$$a_n = L$$

for all $n \geq N$. For example,

$$5, 4, 3, 2, 1, 1, 1, 1, \ldots$$

is eventually constant. Understanding eventually constant sequences is easy, as they are essentially finite lists of numbers.

While not too many sequences are eventually constant, there are plenty of sequences that resemble eventually-constant sequences. Take $b_n = \frac{1}{n}$ $(n \geq 1)$, for example. Suppose for a minute that the smallest unit of distance on the number line is $\frac{1}{10}$. This means tha if two numbers are, say, $\frac{1}{20}$ apart from each other, then we can say no more than that they are close, viz., two numbers that are $\frac{1}{20}$ apart are essentially indistinguishable. With this in mind, all terms in $(b_n)_{n=1}^\infty$ beyond the tenth term b_{10} are essentially indistinguishable from 0: indeed, if $n > 10$, then $\frac{1}{n} < \frac{1}{10}$, and so

$$|b_n - 0| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{10}$$

whenever $n > 10$. In other words, $(b_n)_{n=1}^\infty$ is essentially eventually constant, given the restriction on our measurement system.

Let us suppose that we managed to improve our measurement technology, and that the smallest unit of distance is now $\frac{1}{100}$. It is nevertheless still true that $(b_n)_{n=1}^\infty$ is essentially eventually constant. Indeed, if $n > 100$, then $\frac{1}{n} < \frac{1}{100}$, and so

$$|b_n - 0| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{100}$$

whenever $n > 100$. In other words, b_n is essentially indistinguishable from 0 once $n > 100$.

If we could argue that b_n for large enough n is indistinguishable from 0 for any choice of minimal unit of measurement ε, then it would make sense to say that the sequence $(b_n)_{n=1}^\infty$ *approaches* 0. We formalize this notion below.

Definition 3.1. A sequence $(a_n)_{n=1}^\infty$ is said to *converge* to a number L if, for each choice of $\varepsilon > 0$, there exists an $N > 0$ such that $n \geq N$ implies

$$|a_n - L| < \varepsilon.$$

We say that $(a_n)_{n=1}^\infty$ *diverges* if $(a_n)_{n=1}^\infty$ does not converge to L for any choice of L.

We write $a_n \to L$ to denote that $(a_n)_{n=1}^\infty$ converges to L. Similarly, we write $a_n \not\to L$ to denote that $(a_n)_{n=1}^\infty$ does not converge to L.

Example 3.2. Let $(a_n)_{n=1}^\infty$ be an eventually constant sequence. We can then find a real number L and a positive integer N such that $a_n = L$ for all $n \geq N$.

We claim that $a_n \to L$. To see this, we note that

$$|a_n - L| = |L - L| = 0$$

whenever $n \geq N$. Therefore, regardless of our choice of $\varepsilon > 0$, we have

$$|a_n - L| < \varepsilon$$

for all $n \geq N$. It follows that $a_n \to L$.

We show that $a_n \not\to C$ if $C \neq L$. To see this, we let

$$\varepsilon = \frac{|L - C|}{3},$$

so that $\varepsilon > 0$ and $|L - C| > \varepsilon$. Now, if $n \geq N$, then

$$|a_n - C| = |L - C| > \varepsilon.$$

It follows that there is at least one ε such that no N guarantees

$$|a_n - C| < \varepsilon$$

for all $n \geq N$. We conclude that $a_n \not\to C$. □

Example 3.3. Let $a_n = \frac{1}{n}$ for each $n \geq 1$. We shall show that $a_n \to L$ if $L = 0$, and that $a_n \not\to L$ if $L \neq 0$.

We suppose for now that $L = 0$. Fix $\varepsilon > 0$. We need to find an N such that

$$|a_n - L| < \varepsilon \tag{3.4}$$

for all $n \geq N$. Since

$$|a_n - L| = \left|\frac{1}{n} - 0\right| = \frac{1}{n},$$

(3.4) is equivalent to

$$\frac{1}{n} < \varepsilon,$$

which, in turn, is equivalent to

$$\frac{1}{\varepsilon} < n. \tag{3.5}$$

It thus suffices to find an N such that (3.5) holds for all $n \geq N$. To this end, we recall that $\lfloor x \rfloor$ is the greatest integer smaller than or equal to x. We pick

$$N = \left\lfloor \frac{10}{\varepsilon} \right\rfloor + 1.$$

Since

$$\left\lfloor \frac{10}{\varepsilon} \right\rfloor + 1 > \frac{10}{\varepsilon} > \frac{1}{\varepsilon},$$

we see that $N > 1/\varepsilon$. Therefore, if $n \geq N$, then

$$n \geq N > \frac{1}{\varepsilon},$$

and (3.5) holds. Since our choice of ε was arbitrary, we conclude that $a_n \to 0$.

We now suppose that $L \neq 0$. We set

$$\varepsilon = \frac{|L - 0|}{17},$$

so that $\varepsilon > 0$ and

$$\frac{|L - 0|}{2} > \varepsilon. \tag{3.6}$$

Since $a_n \to 0$, we can find a positive integer N such that $n \geq N$ implies

$$|a_n - 0| < \varepsilon. \tag{3.7}$$

If $n \geq N$, then the reverse triangle inequality (Theorem 2.3) implies that

$$|a_n - L| = |(a_n - 0) - (L - 0)| = |(L - 0) - (a_n - 0)| \geq |L - 0| - |a_n - 0|.$$

Now, for each $n \geq N$, (3.6) implies that

$$|L - 0| - |a_n - 0| > 2\varepsilon - |a_n - 0|,$$

and (3.7) implies that

$$2\varepsilon - |a_n - 0| > 2\varepsilon - \varepsilon = \varepsilon.$$

It follows that

$$|a_n - L| > \varepsilon$$

for all $n \geq N$, whence $a_n \not\to L$. □

Exercise 3.8. Let $b_n = \frac{(-1)^n}{n}$ for each $n \geq 1$. Show that $b_n \to L$ if $L = 0$ and $b_n \not\to L$ if $L \neq 0$.

Exercise 3.9. Let $c_n = \frac{1}{n^2+1}$ for each $n \geq 1$. Show that $c_n \to L$ if $L = 0$ and $c_n \not\to L$ if $L \neq 0$.

Exercise 3.10. Let $d_n = \frac{2n}{n^2+1}$ for each $n \geq 1$. Show that $d_n \to L$ if $L = 0$ and $c_n \not\to L$ if $L \neq 0$.

Exercise 3.11. Fix $p > 0$. Let $e_n = \frac{1}{n^p}$ for each $n \geq 1$. Show that $e_n \to L$ if $L = 0$ and $e_n \not\to L$ if $L \neq 0$.

The above computations suggest that if the limit exists, then it ought to be unique. The following theorem formalizes this observation.

Theorem 3.12 (Uniqueness of limit)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence. If $a_n \to L_1$ and $a_n \to L_2$, then $L_1 = L_2$.*

To establish the above theorem, we need a preliminary result, which states that two numbers that are essentially indistinguishable from each other are, in fact, equal.

Lemma 3.13. *Let a and b be real numbers. If*

$$|a - b| < \varepsilon$$

for each choice of $\varepsilon > 0$, then $a = b$.

Proof of lemma. It suffices to show that $|a - b| = 0$. Indeed, in this case, $a - b = 0$, and so $a = b$.

We now suppose for a contradiction that $|a - b| > 0$. Then we can find a number ε_0 such that

$$0 < \varepsilon_0 < |a - b|,$$

which contradicts the assumption that

$$|a - b| > \varepsilon$$

for all choices of $\varepsilon > 0$. Therefore, the assumption that $|a - b| > 0$ was erroneous, whence we must have $|a - b| = 0$, as desired. □

Proof of the uniqueness theorem. Suppose that $a_n \to L_1$ and $a_n \to L_2$. Fix $\varepsilon > 0$. We can find a positive N_1 such that

$$|a_n - L_1| < \frac{\varepsilon}{2} \tag{3.14}$$

for all $n \geq N_1$. We can also find a positive integer N_2 such that

$$|a_n - L_2| < \frac{\varepsilon}{2} \tag{3.15}$$

for all $n \geq N_2$.

We now set $N = \max(N_1, N_2)$ and observe that, for each $n \geq N$, the triangle inequality (Theorem 2.1) implies that

$$|L_1 - L_2| = |(L_1 - a_n) + (a_n - L_2)| \leq |a_n - L_1| + |a_n - L_2|.$$

It now follows from (3.14) and (3.15) that

$$|a_n - L_1| + |a_n - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We conclude that

$$|L_1 - L_2| < \varepsilon$$

for an arbitrary choice of $\varepsilon > 0$, whence Lemma 3.13 implies that $L_1 = L_2$, as was to be shown. □

Since there is at most one limit, we use the notation

$$\lim_{n\to\infty} a_n$$

to denote *the* limit of $(a_n)_{n=1}^{\infty}$, provided that it exists.

We now consider two sequences that do not have a limit.

Example 3.16. Let $a_n = n$ for all $n \geq 1$. We show that $a_n \not\to L$ for any choice of L.

Fix L. There exists an integer N such that $N > L$. We set

$$\varepsilon = \frac{N - L}{26345234532},$$

so that $\varepsilon > 0$ and $N - L > \varepsilon$.

Now, if $n \geq N$, then $n \geq N > L$, and so

$$|a_n - L| = |n - L| = n - L \geq N - L > \varepsilon.$$

It follows that $a_n \not\to L$. □

Example 3.17. Let $b_n = (-1)^n$ for all $n \geq 1$. We show that $a_n \not\to L$ for any choice of L. If $|L| \neq 1$, then we set

$$\varepsilon = \frac{|1 - |L||}{313562354},$$

so that $\varepsilon > 0$ and $|1 - |L|| > \varepsilon$. The reverse triangle inequality implies that

$$|b_n - L| \geq ||b_n| - |L|| = |1 - |L||$$

for all $n \geq 1$. Since $|1 - |L|| > \varepsilon$, we conclude that

$$|b_n - L| > \varepsilon$$

for all $n \geq 1$. It follows that $b_n \not\to L$ whenever $|L| \neq 1$.

If $L = 1$, then

$$|b_{2n-1} - L| = |-1 - 1| = 2$$

for all $n \geq 1$. Certainly, $b_n \not\to 1$. Similarly, if $L = -1$, then

$$|b_{2n} - L| = |1 - (-1)| = 2$$

for all $n \geq 1$, and so $b_n \not\to -1$. □

While both of the above examples concern divergent sequences, they are of different types. The first sequence continues to increase, whereas the second sequence stays within a bounded interval. The former type has a name:

Definition 3.18. Let $(a_n)_{n=1}^{\infty}$ be a sequence. We say that $(a_n)_{n=1}^{\infty}$ *diverges to infinity* if, for each choice of $M > 0$, there exists a positive integer N such that $n \geq N$ implies

$$a_n > M.$$

We write

$$\lim_{n\to\infty} a_n = \infty$$

to denote that $(a_n)_{n=1}^{\infty}$ diverges to infinity.

Exercise 3.19. What should be the definition of a sequence *diverging to negative infinity*? Discuss.

Exercise 3.20. Determine whether the following divergent sequences diverge to infinity, diverge to negative infinity, or neither.

(1) $a_n = 2^n$
(2) $b_n = -2^n$
(3) $c_n = (-2)^n$

4. Computation of Limits

While Definition 3.1 is perhaps one of the most important concepts in calculus, appealing to it directly is often cumbersome.

One way to circumvent this difficulty is to relate the problem of the computation of the limit of a sequence to the related problem of the computation of the limit of a function. To this end, we recall the definition of the limit of a function:

Definition 4.1. Let f be a function on the real line. We write

$$\lim_{x\to\infty} f(x) = L$$

if, for each choice of $\varepsilon > 0$, there exists a positive real number M such that $x > M$ implies

$$|f(x) - L| < \varepsilon.$$

We write

$$\lim_{x\to\infty} f(x) = \infty$$

if, for each L, there exists a positive real number M such that $x > M$ implies

$$f(x) > L.$$

We write

$$\lim_{x\to\infty} f(x) = -\infty$$

if, for each L, there exists a positive real number M such that $x > M$ implies

$$f(x) < L.$$

Since Definition 4.1 and Definition 3.1 are quite similar, it is reasonable to conjecture the following:

Theorem 4.2. *Let f be a function on the real line. If a sequence $(a_n)_{n=1}^\infty$ satisfies the identity*

$$f(n) = a_n$$

for all $n \geq 1$, then

$$\lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n.$$

provided that $(a_n)_{n=1}^\infty$ either converges, diverges to infinity, or diverges to negative infinity.

We omit the formal proof, which essentially consists of comparing the two analogous definitions of limit carefully.

With Theorem 4.2 at hand, we are now able to invoke the usual tools for the computation of functional limits to compute sequential limit. Of particular note is L'Hôpital's rule.

Example 4.3. Fix $p > 0$. Theorem 4.2 implies that

$$\lim_{n\to\infty} \frac{\ln n}{n^p} = \lim_{x\to\infty} \frac{\ln x}{x^p}.$$

We now invoke L'Hôpital's rule to conclude that

$$\lim_{n\to\infty} \frac{\ln n}{n^p} = \lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0.$$

□

We now recall that a function f is *continuous* if, intuitively, $f(x)$ does not change too much when we change the value x by a little bit. Since our intuition behind the statement

$$\lim_{n\to\infty} a_n = L$$

was that a_n for large values of n are essentially indistinguishable from L, we expect $f(a_n)$ for large values of n to be essentially indistinguishable from $f(L)$, given our characterization of continuous functions. This leads to the following sequential characterization of continuity:

Theorem 4.4 (Sequential characterization of continuity)**.** *f is a continuous function if and only if*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

whenever $(a_n)_{n=1}^\infty$ is a convergent sequence.

Once again, we omit the formal proof of this theorem, which is beyond the scope of this course.

Example 4.5. We wish to compute the sequential limit

$$\lim_{n\to\infty} n^{1/n}.$$

By Example 4.3,

$$\lim_{n\to\infty} \ln n^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Observe now that

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln n^{1/n}}.$$

Since the exponential function is continuous, it now follows from the sequential characterization of continuity (Theorem 4.4) that

$$\lim_{n\to\infty} e^{\ln n^{1/n}} e^{\lim_{n\to\infty} \ln n^{1/n}} = e^0 = 1.$$

□

An extremely useful tool in the study of functional limits is the *algebraic limit theorem*, which tells us how to compute the limits of sums, differences, products, quotients, and scalar multiples of functions. Given the apparent connections between functional limits and sequential limits, we expect the following theorem to hold as well:

Theorem 4.6 (Algebraic limit theorem)**.** *Suppose that $a_n \to a$ and $b_n \to b$.*

(1) $ca_n \to ca$ for each choice of c;
(2) $a_n + b_n \to a + b$;
(3) $a_n - b_n \to a - b$;
(4) $a_n b_n \to ab$;
(5) $a_n/b_n \to ab$, provided that $b \neq 0$ and $b_n \neq 0$ for all $n \geq 1$;
(6) $a_n^p \to a^p$ for each choice of $p > 0$.

Example 4.7. We wish to compute

$$\lim_{n\to\infty} \frac{n}{n+1}.$$

Since

$$\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}},$$

it follows from the algebraic limit theorem that

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \frac{1}{n}} = \frac{1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} = \frac{1}{1} = 1.$$

□

We caution the reader that the convergence of both $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ in the algebraic limit theorem is essential. In fact, pathological behaviors arise when we deal with divergent sequences.

Example 4.8. Let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ for each $n \geq 1$. We know that both diverge; nevertheless, $a_n + b_n = 0$ for all $n \geq 1$, and so $a_n + b_n \to 0$. □

Example 4.9. Let $a_n = \frac{1}{n}$ and $b_n = n$ for each $n \geq 1$, so that $a_n \to 0$ and $(b_n)_{n=1}^\infty$ diverges. $a_n b_n = 1$ for all $n \geq 1$, and so $a_n b_n \to 1$. □

Example 4.10. Let $a_n = 1$ and $b_n = 1/n$ for each $n \geq 1$, so that $a_n \to 1$ and $b_n \to 0$. Since $a_n/b_n = n$ for all $n \geq 1$, we see that $(a_n/b_n)_{n=1}^\infty$ diverges.

Exercise 4.11. Find sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ such that $(a_n)_{n=1}^\infty$ converges, $(b_n)_{n=1}^\infty$ diverges, and $(a_n + b_n)_{n=1}^\infty$ converges.

Another extremely useful tool for computing functional limits is the *squeeze theorem*. Let us now establish its sequential analogue.

Theorem 4.12 (Squeeze theorem). *Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, and $(c_n)_{n=1}^{\infty}$ be sequences. If $a_n \leq b_n \leq c_n$ for all $n \geq 1$ and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $(b_n)_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} b_n = L.$$

Proof. Fix $\varepsilon > 0$. We find a positive integer N_1 such that $n \geq N_1$ implies

$$|a_n - L| < \varepsilon,$$

so that

$$-\varepsilon \leq a_n - L. \tag{4.13}$$

We also find another positive integer N_2 such that $n \geq N_2$ implies

$$|c_n - L| < \varepsilon,$$

so that

$$c_n - L < \varepsilon. \tag{4.14}$$

Set $N = \max(N_1, N_2)$. For each $n \geq N$, (4.13), (4.14), and the assumption

$$a_n \leq b_n \leq c_n$$

together imply that

$$-\varepsilon < a_n - L \leq b_n - L \leq c_n - L < \varepsilon.$$

It follows that

$$-\varepsilon < b_n - L < \varepsilon$$

for all such n, whence

$$|b_n - L| < \varepsilon.$$

It follows that $b_n \to L$. □

Example 4.15. We wish to compute

$$\lim_{n\to\infty} \frac{n!}{n^n}.$$

Observe first that $n! \geq 0$ and $n^n \geq 0$ for all $n \geq 1$, and so

$$\frac{n!}{n^n} \geq 0$$

for all $n \geq 1$. Moreover,

$$\frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n} \leq \frac{1}{n} \cdot 1 \cdot 1 \cdots 1 = \frac{1}{n},$$

and so

$$\frac{n!}{n^n} \leq \frac{1}{n}$$

for all $n \geq 1$. Since

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0,$$

it now follows from the Squeeze Theorem (Theorem 4.12) that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

□

Exercise 4.16. We define the *double factorial* as follows:

- $1!! = 1$
- $2!! = 2$
- $3!! = 3 \cdot 1$
- $4!! = 4 \cdot 2$
- $5!! = 5 \cdot 3 \cdot 1$
- $6!! = 6 \cdot 4 \cdot 2$
- $7!! = 7 \cdot 5 \cdot 3 \cdot 1$
- and so on

Verify the inequality

$$\frac{n}{n+1} < \sqrt{\frac{n}{n+2}}$$

for all $n \geq 1$, and use the inequality to show that

$$\lim_{n \to \infty} \frac{(n+1)!!}{n!!} = 0.$$

Exercise 4.17. Show that if $|a_n| \to 0$, then $a_n \to 0$.

Exercise 4.18. Show that

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < r; \\ 1 & \text{if } r = 1. \end{cases}$$

The squeeze theorem can also be thought of as a manifestation of a general ordering principle, established below:

Theorem 4.19 (Order limit theorem)**.** *Suppose that $a_n \to a$ and $b_n \to b$. If $a_n \leq b_n$ for each $n \geq 1$, then $a \leq b$.*

Proof of lemma. Let $c_n = b_n - a_n$, so that $c_n \geq 0$ for all $n \geq 1$. By the algebraic limit theorem (Theorem 4.6), we have

$$\lim_{n \to \infty} c_n = b - a,$$

whence it suffices to show that

$$\lim_{n \to \infty} c_n \geq 0.$$

For notational convenience, let us set

$$c = \lim_{n \to \infty} c_n.$$

We assume for a contradiction that $c < 0$. Set

$$\varepsilon = \frac{-c}{462345},$$

so that

$$-c > \varepsilon.$$

Since $c_n \to c$, we can find a positive integer N such that

$$|c_n - c| < \frac{\varepsilon}{2}.$$

for all $n \geq N$. Therefore, for all such n,

$$c_n - c < \frac{\varepsilon}{2},$$

and so

$$c_n < \frac{\varepsilon}{2} + c < \frac{\varepsilon}{2} - \varepsilon = 0\frac{\varepsilon}{2} < 0.$$

It follows that $c_n < 0$ for all $n \geq N$, which contradicts the assumption that $c_n \geq 0$ for all $n \geq 1$. We conclude that $c \geq 0$. □

Exercise 4.20. Suppose that $a_n \to a$, $b_n \to b$, $c_n \to c$,

$$a_n \leq b_n \leq c_n$$

for all $n \geq 1$, and that $a = c$. Show that

$$a = b = c$$

using the order limit theorem.

5. Monotone Convergence Theorem

In the final section, we develop a theoretical tool that plays a significant role in the study of *series*, which we define now.

Definition 5.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence. For each positive integer N, we define the *Nth partial sum* of $(a_n)_{n=1}^{\infty}$ is defined to be

$$S_N = \sum_{n=1}^{N} a_n = a_1 + \cdots + a_N.$$

The *sum* of $(a_n)_{n=1}^{\infty}$ is defined to be the limit

$$\lim_{N \to \infty} S_N.$$

We write

$$\sum_{n=1}^{\infty} a_n$$

to denote the sum of $(a_n)_{n=1}^{\infty}$; the expression is also referred to as the *series* corresponding to the sequence $(a_n)_{n=1}^{\infty}$.

Although it might not make sense at first that an infinite sum of positive numbers such as

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^N} + \cdots$$

can be finite, the following example shows that it is possible.

Example 5.2 (Geometric series). Fix $-1 < r < 1$ and a real number a. We shall show that

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}.$$

To this end, we observe that

$$S_N = \sum_{n=1}^{N} ar^n = ar + \sum_{n=2}^{N} ar^n$$

$$rS_N = \sum_{n=1}^{N} ar^{n+1} = \sum_{n=2}^{N+1} ar^n = \sum_{n=2}^{N} ar^n + ar^{N+1}.$$

Therefore,

$$(1-r)S_N = ar + \left(\sum_{n=2}^{N} ar^n - \sum_{n=2}^{N} ar^n\right) + ar^{N+1} = a(r - r^{N+1}).$$

It then follows from Exercise 4.18 and the Algebraic Limit Theorem (Theorem 4.6) that

$$\sum_{n=1}^{\infty} ar^n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(r-r^{N+1})}{1-r} = \frac{ar}{1-r},$$

as was to be shown. □

We may now ask ourselves: when is an infinite sum of positive numbers finite? To answer this question, we note that the partial sums of a series consisting entirely of positive summands form an *increasing sequence*, which we now define:

Definition 5.3. Let $(a_n)_{n=1}^{\infty}$ be a sequence. We say that $(a_n)_{n=1}^{\infty}$ is *increasing* if

$$a_n \le a_{n+1}$$

for all $n \ge 1$, and *decreasing* if

$$a_n \ge a_{n+1}$$

for all $n \ge 1$. $(a_n)_{n=1}^{\infty}$ is said to be *monotone* if it is either increasing or decreasing.

If the partial sums grow without bound, then the infinite sum most definitely will not be finite. We therefore require that the sequence remains *bounded*:

Definition 5.4. Let $(a_n)_{n=1}^{\infty}$ be a sequence. We say that $(a_n)_{n=1}^{\infty}$ is *bounded above* if there exists a number M such that

$$a_n \le M$$

for all $n \ge 1$, and *bounded below* if there exists a number m such that

$$a_n \ge m$$

for all $n \ge 1$. $(a_n)_{n=1}^{\infty}$ is said to be *bounded* if it is bounded above and below.

It turns out that these two conditions are sufficient to guarantee convergence.

Theorem 5.5 (Monotone convergence theorem)**.** *Every bounded monotone sequence converges.*

We omit the proof of the theorem, which makes use of a crucial property of the real numbers known as the *least upper bound property.*

Corollary 5.6. *If $(a_n)_{n=1}^{\infty}$ is a sequence of positive numbers, and if there exists a number M such that*

$$\sum_{n=1}^{N} a_n \le M,$$

then

$$\sum_{n=1}^{\infty} a_n$$

exists and is bounded above by M.

Proof. Since $a_n \geq 0$ for all $n \geq 1$, we see that the partial sums

$$S_N = \sum_{n=1}^{N} a_n$$

form a nonnegative, monotone sequence. Since $0 \leq S_N \leq M$ for all $N \geq 1$ $(S_N)_{N=1}^{\infty}$ is bounded. Since $(S_N)_{N=1}^{\infty}$ is a bounded monotone sequence, the monotone convergence theorem (Theorem 5.5) implies that the limit

$$\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} a_n$$

exists. It now follows from the Order Limit Theorem (Theorem 4.19) that

$$\sum_{n=1}^{\infty} a_n \leq M,$$

as was to be shown. □