CALCULUS II, SUMMER 2015 - DIRICHLET TEST

We recall the following theorem from class:

Theorem 1 (Dirichlet test)**.** *Let* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be sequences such that*

(1) $a_n \geq 0$ *for each* $n \in \mathbb{N}$*;*
(2) $a_n \geq a_{n+1}$ *for each* $n \in \mathbb{N}$*;*
(3) $a_n \to 0$*;*
(4) *there exists an* $M > 0$ *such that* $\left|\sum_{n=1}^{N} b_n\right| \leq M$ *for all* $N \in \mathbb{N}$*.*

Then

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

We write $a_n \searrow 0$ to denote that $(a_n)_{n=1}^{\infty}$ satisfies the conditions (1), (2), and (3).

Example 2. We show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges. To this end, we set $a_n = \frac{1}{n}$ and $b_n = (-1)^{n+1}$. Then $a_n \searrow 0$ and

$$\left|\sum_{n=1}^{N} b_n\right| = \left|\sum_{n=1}^{N} (-1)^{n+1}\right| \leq 1$$

for all $N \in \mathbb{N}$. Therefore, the Dirichlet test implies that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges.

The special case of Dirichlet test with $b_n = (-1)^{n+1}$ is often useful:

Theorem 3 (Alternating series test)**.** *If* $a_n \searrow 0$*, then*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges. □

Example 4. We show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 + 1}$$

converges. To this end, we let

$$a_n = \frac{n}{n^2 + 1}.$$

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By the alternating series test, it suffices to show that $a_n \searrow 0$. Evidently, $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \to 0$. It is thus enough to prove that $(a_n)_{n=1}^{\infty}$ is decreasing.

Observe that the statement

$$a_n \geq a_{n+1}$$

is equivalent to

$$n[(n+1)^2+1] \geq (n+1)(n^2+1)$$

This, in turn, is equivalent to

$$n^3+2n^2+2n \geq n^3+n^2+n+1.$$

This inequality is equivalent to

$$n^2+n \geq 1,$$

which we know to be true for all $n \in \mathbb{N}$. We conclude that

$$a_n \geq a_{n+1}$$

for all $n \in \mathbb{N}$, whence it now follows from the alternating series test that

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}$$

converges. □

Example 5. We show that

$$\sum_{n=1}^{\infty}(-1)^n\left(\arctan(-n)+\frac{\pi}{2}\right)$$

converges. To this end, we set

$$f(x) = \arctan(-x)+\frac{\pi}{2}$$

and note that

$$f'(x) = -\frac{1}{x^2+1}.$$

Since $f'(x) < 0$ for all x, it follows that $(f(n))_{n=1}^{\infty}$ is a decreasing sequence.

We now observe that

$$\lim_{x\to\infty} f(x) = -\frac{\pi}{2}+\frac{\pi}{2} = 0,$$

and so $f(n) \searrow 0$. It now follows from the alternating series test that

$$\sum_{n=1}^{\infty}(-1)^n\left(\arctan(-n)+\frac{\pi}{2}\right)$$

converges. □

Example 6. We show that

$$\sum_{n=1}^{\infty}\cos(n\pi)\arctan(-n)+\frac{\pi\sin((n+1/2)\pi)}{2}$$

converges. Since $\cos(n\pi) = (-1)^{n+1}$ and $\sin((n+1/2)\pi) = (-1)^{n+1}$, we see that

$$\sum_{n=1}^{\infty}\cos(n\pi)\arctan(-n)+\frac{\pi\sin((n+1/2)\pi)}{2} = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\arctan(-n)+\frac{\pi}{2}\right),$$

which we already know the convergence of. □

Example 7. We show that

$$\sum_{n=1}^{\infty} \frac{1+(-1)^n n}{n^2}$$

converges. Observe that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by the p-test. Moreover,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges by the alternating series test. It now follows from the algebraic limit theorem that

$$\sum_{n=1}^{\infty} \frac{1+(-1)^n n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2}$$

converges. □

Example 8. We show that

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n}$$

converges. Since $\frac{1}{n} \searrow 0$, the Dirichlet test implies that it suffices to show that there exists an $M > 0$ such that

$$\left|\sum_{n=1}^{N} \sin(n\pi/2)\right| \leq M$$

for each $N \in \mathbb{N}$.

To this end, we compute the first several partial sums:

$$\sum_{n=1}^{1} \sin(n\pi/2) = 1;$$
$$\sum_{n=1}^{2} \sin(n\pi/2) = 1 + 0 = 1;$$
$$\sum_{n=1}^{3} \sin(n\pi/2) = 1 + 0 - 1 = 0;$$
$$\sum_{n=1}^{4} \sin(n\pi/2) = 1 + 0 - 1 + 0 = 0;$$
$$\sum_{n=1}^{5} \sin(n\pi/2) = 1 + 0 - 1 + 0 + 1 = 1.$$

Since sin is 2π-periodic, we see, in general, that

$$\sum_{n=1}^{4p+q} \sin(n\pi/2) = \left(p \sum_{n=1}^{4} \sin(n\pi/2)\right) + \sum_{n=1}^{q} \sin(n\pi/2) = \sum_{n=1}^{q} \sin(n\pi/2)$$

for all $p \in \mathbb{N}$ and $0 \le q \le 3$. It thus follows that

$$\left|\sum_{n=1}^{N} \sin(n\pi/2)\right| \le 1$$

for all choices of $N \in \mathbb{N}$. We conclude from the Dirichlet test that

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n}$$

converges.

Example 9. We show that

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}$$

converges. Since $\frac{1}{n} \searrow 0$, the Dirichlet test implies that it suffices to show that there exists an $M > 0$ such that

$$\left|\sum_{n=1}^{N} \sin n\right| \le M$$

for all $N \in \mathbb{N}$.

To this end, we recall that

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B).$$

Using this identity, we see that, for any choice of N,

$$\begin{aligned}
&\sum_{n=1}^{N} \sin n \\
=&\frac{1}{2\sin\frac{1}{2}} \sum_{n=1}^{N} 2\sin\frac{1}{2}\sin n = \frac{1}{2\sin\frac{1}{2}} \sum_{n=1}^{N} \cos\left(n - \frac{1}{2}\right) - \cos\left(n + \frac{1}{2}\right) \\
=&\frac{1}{2\sin\frac{1}{2}} \left(\sum_{n=1}^{N} \cos\left(n - \frac{1}{2}\right) - \sum_{n=1}^{N} \cos\left(n + \frac{1}{2}\right)\right) \\
=&\frac{1}{2\sin\frac{1}{2}} \left(\cos\frac{1}{2} - \cos\frac{3}{2} + \cos\frac{3}{2} - \cos\frac{5}{2} + \cdots + \cos\frac{2N-1}{2} - \cos\frac{2N+1}{2}\right) \\
=&\frac{1}{2\sin\frac{1}{2}} \left(\cos\frac{1}{2} - \cos\frac{2N+1}{2}\right).
\end{aligned}$$

Therefore, the triangle inequality implies that

$$\left|\sum_{n=1}^{N} \sin n\right| \le \frac{|\cos\frac{1}{2}| + |\cos\frac{2N+1}{2}|}{2\sin\frac{1}{2}} \le \frac{1+1}{2\sin\frac{1}{2}} = \frac{1}{\sin\frac{1}{2}}.$$

Since the choice of N was arbitrary, the Dirichlet test now implies that

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}$$

converges.

Example 10. We show that if $a_n \searrow 0$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely, then

$$\sum_{n=1}^{\infty} a_n b_n$$

converges. Setting $M = \sum_{n=1}^{\infty} |b_n|$, we see that

$$\left|\sum_{n=1}^{N} b_n\right| \leq \sum_{n=1}^{N} |b_n| \leq \sum_{n=1}^{\infty} |b_n| = M$$

for each $N \in \mathbb{N}$. It follows from the Dirichlet test that

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Example 11. We show that

$$\sum_{n=1}^{\infty} \frac{n^{n+2} n!}{(n^3+1)n^n}$$

converges. We rewrite the above series as follows:

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \cdot \frac{n!}{n^n}.$$

By Example 10, it suffices to show that

$$\frac{n^2}{n^3+1} \searrow 0$$

and that

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges absolutely.

To show that

$$\frac{n^2}{n^3+1} \searrow 0,$$

we set

$$f(x) = \frac{x^2}{x^3+1}$$

and observe that

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}.$$

Note that $x^3 + 1 > 0$ for all $x \geq 1$. Since $2 - x^3 < 0$ for all $x > \sqrt[3]{2}$, we see, in particular, that $2 - x^3 < 0$ for all $x \geq 1$. It follows that

$$f'(x) < 0$$

for all $x \geq 1$. This implies that f is decreasing on $[1, \infty)$, whence, in particular, the sequence

$$a_n = \frac{n^2}{n^3+1}$$

is decreasing. Now, $a_n \geq 0$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0,$$

and so we conclude that $a_n \searrow 0$.

To show that

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges absolutely, we apply the ratio test. Set

$$b_n = \frac{n!}{n^n}$$

for each $n \in \mathbb{N}$ and observe that

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| &= \lim_{n\to\infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n \\ &= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1. \end{aligned}$$

It follows from the ratio test that

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges absolutely.

We now conclude from Example 10 that

$$\sum_{n=1}^{\infty} \frac{n^{n+2} n!}{(n^3+1) n^n}$$

converges. □