CALCULUS II, SUMMER 2015 - BESSEL FUNCTION OF ORDER ZERO

For each nonnegative integer r, we define the *Bessel function of order* r to be

$$J_r(x) = \left(\frac{x}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{x}{2}\right)^{2n}.$$

In this set of notes, we study the $r = 0$ case:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

This is a power series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where $x_0 = 0$, $a_{2n-1} = 0$ for all $n \geq 1$, and

$$a_{2n} = \frac{(-1)^n}{2^{2n}(n!)^2}.$$

1. Radius of Convergence

1.1. **Computation of the Radius of Convergence.** Set

$$b_n(x) = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

for each $n \geq 0$, so that

$$J_0(x) = \sum_{n=0}^{\infty} b_n(x).$$

Observe that, for a fixed $x \in \mathbb{R}$,

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{b_{n+1}(x)}{b_n(x)} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2} / 2^{2n+2}((n+1)!)^2}{(-1)^n x^{2n} / 2^{2n}(n!)^2} \right| \\ &= \lim_{n\to\infty} \left| \frac{-x^2}{2^2(n+1)^2} \right| = 0\end{aligned}$$

regardless of the choice of $|x|$. The ratio test implies that J_0 converges absolutely for all $x \in \mathbb{R}$. □

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1.2. **A Cautionary Remark.** We remark that it is impractical to use the power-series radius of convergence formulas

$$R = \left(\lim_{n\to\infty} \sqrt[n]{|a_n|} \right)^{-1} = \left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \right)^{-1}$$

directly, as the odd terms a_{2n+1} are zero. We can nevertheless use the ratio test or the root test directly to the series

$$\sum_{n=0}^{\infty} b_n(x)$$

for each fixed x.

1.3. **Root Test?** We could, of course, apply the root test to

$$\sum_{n=0}^{\infty} b_n(x)$$

as well. Let's see what we get:

$$\lim_{n\to\infty} \sqrt[n]{|b_n(x)|} = \lim_{n\to\infty} \sqrt[n]{\frac{|(-1)^n x^{2n}|}{2^{2n}(n!)^2}} = \lim_{n\to\infty} \frac{|x|^2}{4(n!)^{2/n}}.$$

It would appear that we need to know how to compute

$$\lim_{n\to\infty} (n!)^{2/n}.$$

To this end, we observe that a simple arrangement yields the following:

$$(n!)^2 = (n \cdot 1)((n-1)\cdot 2)((n-2)\cdot 3)\cdots(3\cdot(n-2))(2\cdot(n-1))(1\cdot n).$$

In other words,

$$(n!)^2 = \prod_{k=1}^{n} (n+1-k)k.$$

Now, for each $1 \leq k \leq n$,

$$(n+1-k)k - n = nk - n + k - k^2 = n(k-1) + k(k-1) = (n+k)(k-1) \geq 0,$$

and so

$$(n+1-k) \geq n.$$

It follows that

$$(n!)^2 = \prod_{k=1}^{n} (n+1-k)k \geq \prod_{k=1}^{n} n = n^n,$$

whence it follows that

$$\lim_{n\to\infty} (n!)^{2/n} \geq \lim_{n\to\infty} n = \infty.$$

Returning to the task at hand, we see that

$$\lim_{n\to\infty} \sqrt[n]{|b_n(x)|} = \lim_{n\to\infty} \frac{|x|^2}{4(n!)^{2/n}} = 0$$

regardless of the choice of x. The root test therefore implies that J_0 converges absolutely for all x.

2. Bessel's Differential Equation

We now show that

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = 0$$

for all $x \in \mathbb{R}$. To this end, we observe that

$$\begin{aligned} J_0'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-1} n!(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2(n+1)-1}}{2^{2(n+1)-1}(n+1)!((n+1)-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2^{2n+1} n!(n+1)!}. \end{aligned}$$

Observe that $J_0'(x)$ does not have a constant term, and so the computation of $J_0''(x)$ does not eliminate the $n = 0$ term:

$$J_0''(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)x^{2n}}{2^{2n+1} n!(n+1)!}.$$

Now, for each $x \in \mathbb{R}$,

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n+1} n!(n+1)!} \left[2(n+1) - 1 - (2n+1)\right] = 0,$$

as was to be shown. □