COMMENTS ON STYLE AND SAMPLE SOLUTIONS

MARK H. KIM

Comments on Style

I will assess the quality of exposition as well as the correctness of the arguments. Mathematical proofs are *bona fide* pieces of writing, and I request that you treat them as such. Please make a serious effort to communicate your ideas efficiently and elegantly. Now, I understand that English is not the first language for some of you, so it does not make sense to take points off for grammar or any other English-specific issues. There is no excuse for sloppiness, however, and points *will* be deducted for poor explanation of correct ideas. Here are some pointers:

(1) As a general rule, you should not submit the first draft. Do the problems on scratch paper and think about how you should organize your proofs before writing them down. In particular, if you wrote down a correct proof in an incorrect order, then your proof would cease to be correct.
(2) Please keep in mind that your proof is wrong by default if I, the grader, cannot understand it. If you're a top expert in your field, then people will try their best to read your works regardless of the quality of presentation. But otherwise, most of us, including myself, have very little motivation to read someone else's incomprehensible proofs.
(3) One way to improve your proof is to employ short sentences and paragraphs. Each paragraph should only contain one main idea; each sentence should only say one thing. Math is difficult enough without having to parse overly complex writing.
(4) To achieve (3), you must, of course, use actual sentences. Please write in complete sentences without shorthand notations like $\forall$, $\exists$, $\Leftarrow$, $\Leftrightarrow$, s.t., WLOG, and so on. No, using logical quantifiers does not make an argument *more mathematical*. And yes, using them does make your proof substantially less readable.
(5) "Clearly" implies that there is a gap in your reasoning, and that you expect the reader to be able to fill it in with no trouble. This means "clearly" is not the right word to use if you are merely restating a definition, a theorem, or a lemma—in this case, there is no gap in your reasoning. It also means that "clearly" is never the right word to use in a homework write-up. If the gap has to do with a fact that everyone in the class is expected to know, then you can simply use it without comment. If not, then you really shouldn't leave that gap unfilled.
(6) Being wordy is not good, either! Include only the necessary details.

The above list is inspired by James Munkres's *Comments on Style*, which you can find easily on the internet. Another quick but useful resource is a little booklet called *How to Write Mathematics*, published by the AMS in 1973. Nicholas J. Higham's *Handbook of Writing for the Mathematical Sciences* is also useful.

Sample Solutions (for Problem Set 1)

Notations: Given a partition $P = \{x_0, \ldots, x_N\}$ of $[a,b]$ and a function $f : [a,b] \to \mathbb{R}$, we let

$$M_n(f) = \max_{x \in [x_{n-1}, x_n]} f(x) \quad \text{and} \quad m_n(f) = \min_{x \in [x_{n-1}, x_n]} f(x).$$

Problem 1. (a) Fix a partition P of $[a,b]$. Since $\|P\| \geq |x_n - x_{n-1}|$ for all $1 \leq n \leq N$, we see that $\omega_f(\|P\|) \geq M_n - m_n$ for all $1 \leq n \leq N$. Therefore,

$$\begin{aligned} U_\alpha(f,P) - L_\alpha(f,P) &= \sum_{n=1}^{N} (M_n(f) - m_n(f))(\alpha(x_n) - \alpha(x_{n-1})) \\ &\leq \sum_{n=1}^{N} \omega_f(\|P\|)(\alpha(x_n) - \alpha(x_{n-1})) \\ &= \omega_f(\|P\|)(\alpha(b) - \alpha(a)), \end{aligned}$$

as was to be shown. □

Remark 1.1. In what sense does this "quantify" the result $\mathscr{C}([a,b]) \subseteq \mathscr{R}_\alpha([a,b])$? Recall that a function $f : [a,b] \to \mathbb{R}$ is *Lipschitz continuous with constant K* if $|f(x) - f(y)| \leq K|x-y|$ for all $x, y \in [a,b]$. This is equivalent to the statement that $\omega_f(\delta) \leq K\delta$ for all $\delta \in [0,\infty)$—please check this for yourselves! Similarly, given a fixed $0 < \alpha \leq 1$, a function $f : [a,b] \to \mathbb{R}$ is *α-Hölder continuous with constant K* if $|f(x) - f(y)| \leq K|x-y|^\alpha$ for all $x, y \in [a,b]$. Again, this is equivalent to the statement that $\omega_f(\delta) \leq K\delta^\alpha$ for all $\delta \in [0,\infty)$.

From what we have just shown above, we see that the upper and lower sums of Lipschitz functions converge faster than, say, those of (1/2)-Hölder functions. Please convince yourselves why this might be true. □

(b) Here α is continuous instead of f, so

$$U_\alpha(f,P) - L_\alpha(f,P) \leq \sum_{n=1}^{N} (M_n - m_n)\omega_\alpha(\|P\|)$$

for each partition P of $[a,b]$. Monotonicity of f implies that

$$\sum_{n=1}^{N} (M_n(f) - m_n(f)) = M_N(f) - m_1(f) \leq |f(b) - f(a)|,$$

and so

$$\sum_{n=1}^{N} (M_n(f) - m_n(f))\omega_\alpha(\|P\|) = (M_N(f) - m_1(f))\omega_\alpha(\|P\|) \leq |f(b) - f(a)|\omega_\alpha(\|P\|).$$

The desired result now follows. □

Problem 2. Recall that

(a) a linear combination of Riemann-Stieltjes-integrable functions is linear, and
(b) the absolute value of a Riemann-Stieltjes-integrable function is Riemann integrable.

Since

$$\max\{f,g\} = \frac{f+g+|f-g|}{2} \qquad \text{and} \qquad \min\{f,g\} = \frac{f+g-|f-g|}{2},$$

the desired result follows immediately from (a) and (b). ☐

Remark 2.1. We could, of course, take two partitions—one for f, another for g—and consider their common refinement.

Problem 3. Consider, for example, $\alpha(x) = x$ and

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a,b] \\ -1 & \text{if } x \in [a,b] \smallsetminus \mathbb{Q}. \end{cases}$$

Note that $|f| \equiv 1$. Since constant functions are Riemann-integrable, $|f| \in \mathscr{R}_\alpha([a,b])$.

On the other hand, we claim that $f \notin \mathscr{R}_\alpha([a,b])$. To see this, we fix an arbitrary partition P. Since the rationals and the irrationals are dense in $\mathbb{R}$, respectively, we see that $U(f,P) = 1$ and $L(f,P) = -1$. The claim now follows from Riemann's condition. ☐

Problem 4. Fix a partition $P = \{x_0, \ldots, x_N\}$ and let $I_n = [x_{n-1}, x_n]$ for each $1 \leq n \leq N$. We examine three cases:

(1) If $f \geq 0$ on I_n, then $f \equiv |f|$ on I_n, and so $M_n(f) = M_n(|f|)$ and $m_n(f) = m_n(|f|)$.
(2) If $f \leq 0$ on I_n, then $f \equiv -|f|$ on I_n, and so $M_n(f) = -m_n(|f|)$ and $m_n(f) = -M_n(|f|)$.
(3) If f changes sign on I_n, then $M_n(f) = M_n(|f|)$ and $m_n(f) \leq 0 \leq m_n(|f|)$. Therefore, $M_n(f) - m_n(f) \geq M_n(|f|) - m_n(|f|)$.

In all cases, we see that

$$M_n(f) - m_n(f) \geq M_n(|f|) - m_n(|f|).$$

Since n was arbitrary, the desired result now follows from Riemann's condition. ☐

Problem 5. `A pre-proof remark: I am pretty sure` $[a,b]$ `is supposed to be` $[0,1]$ `for this problem. The proof below takes this assumption for granted.`

Let $p_1, \ldots, p_N$ be the accumulation points of Ω. Fix $\varepsilon > 0$ and let $I_n = [p_n - \varepsilon/4N, p_n + \varepsilon/4N] \cap [0,1]$ for each $1 \leq n \leq N$. By the definition of accumulation points, $\Omega' = \Omega \smallsetminus \left(\bigcup_{n=1}^N I_n\right)$ is a finite set. We assume without loss of generality that Ω' is nonempty and write $\{q_1, \ldots, q_M\}$ to represent this set.

We define a partition P of $[0,1]$ as follows:

(1) pick subintervals $J_1, \ldots, J_N$ of $[0,1]$ such that each J_n is of length at most $\varepsilon/2N$, $p_n \in J_n$, and
$$\bigcup_{n=1}^N J_n = \bigcup_{n=1}^N I_n;$$
(2) pick subintervals $J_1', \ldots, J_M'$ such that each J_m' is of length at most $\varepsilon/2M$ and $q_m \in J_m'$;
(3) let $J = J_1 \cup \cdots \cup J_N \cup J_1' \cup \cdots \cup J_M'$ and divide $[0,1] \smallsetminus J$ into finitely many subintervals of arbitrary length;

consolidating the endpoints of the subintervals of type (1), (2), and (3), we obtain a partition P of $[0,1]$.

Observe that there are at most N intervals of type (1), and each yields the following estimate

$$\max_{x\in J_n}\chi_\Omega(x)-\min_{x\in J_n}\chi_\Omega(x)\leq 1\cdot\mathrm{length}(J_n)\leq\frac{\varepsilon}{2N}.$$

Similarly, there are at most M intervals of type (2), and each yields the following estimate:

$$\max_{x\in J'_m}\chi_\Omega(x)-\min_{x\in J'_m}\chi_\Omega(x)\leq 1\cdot\mathrm{length}(J'_m)\leq\frac{\varepsilon}{2M}.$$

For (3), we note that χ_Ω is uniformly zero on $[0,1]\smallsetminus J$, whence

$$\max_{x\in I}\chi_\Omega(x)-\min_{x\in I}\chi_\Omega(x)=0$$

whenever I is a subinterval of type (3).

Combining the above observations, we conclude that

$$U(\chi_\Omega,P)-L(\chi_\Omega,P)\leq\frac{\varepsilon}{2N}\cdot N+\frac{\varepsilon}{2M}\cdot M=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since ε was arbitrary, we conclude from Riemann's condition that χ_Ω is integrable. □

Sample Solutions (for Problem Set 2)

Problem 1. By compactness, we can find $x_0\in[a,b]$ such that $f(x_0)=M$. Fix $\varepsilon>0$ and invoke the continuity of f to find $\delta>0$ such that $f(x)\geq M-\varepsilon$ for all $x\in[x_0-\delta,x_0+\delta]\cap[a,b]$. Find $c,d\in\mathbb{R}$ such that $[c,d]=[x_0-\delta,x_0+\delta]\cap[a,b]$. Positivity of f on $[a,b]$ now implies that

$$\begin{aligned}\left[\int_a^b f(x)^n\,dx\right]^{1/n} &\geq \left[\int_c^d f(x)^n\,dx\right]^{1/n}\\ &> \left[\int_c^d (M-\varepsilon)^n\,dx\right]^{1/n}\\ &= (M-\varepsilon)(d-c)^{1/n},\end{aligned}$$

whence sending $n\to\infty$ yields the lower bound

$$\lim_{n\to\infty}\left[\int_a^b f(x)^n\,dx\right]^{1/n}\geq M-\varepsilon.$$

Since ε was arbitrary, we conclude that

$$\lim_{n\to\infty}\left[\int_a^b f(x)^n\,dx\right]^{1/n}\geq M.$$

We now claim that the lower bound is optimal, i.e.,

$$\lim_{n\to\infty}\left[\int_a^b f(x)^n\,dx\right]^{1/n}=M.$$

To establish the upper bound, we invoke the monotonicity of the Riemann integrabl to conclude that

$$\left[\int_a^b f(x)^n \, dx\right]^{1/n} \leq \left[\int_a^b M^n \, dx\right]^{1/n} M(b-a)^{1/n}.$$

Sending $n \to \infty$, we see that

$$\lim_{n\to\infty} \left[\int_a^b f(x)^n \, dx\right]^{1/n} \leq M,$$

as was to be shown. □

Problem 2. For each $n \in \mathbb{N}$, we define a partition $P_n = \{x_0, \ldots, x_n\}$ of $[0,1]$ and a set of points $T = \{x_1, \ldots, x_n\}$ by setting $x_k = k/n$ for each $0 \leq k \leq n$. Observe that $\|P_n\| = 1/n$, and that

$$S(f, P_n, T_n) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n}. \tag{1}$$

Since $\|P_n\| \to 0$ as $n \to \infty$, the desired result now follows from Corollary 52.6 in the textbook. □

Problem 3. Since f is Lipschitz with constant C,

$$\omega_f(\delta) \leq C\delta$$

for all $\delta \in [0, \infty)$, where ω_f is the modulus of continuity of f: see Remark 1.1 from Homework 1. Homework 1, Problem 1(a) implies that

$$U(f, P) - L(f, P) \leq C\|P\| \tag{2}$$

for each partition P of $[0,1]$.

We now let P_n and T_n denote the sets defined in Problem 2. Fix $n \in \mathbb{N}$. Since

$$L(f, P_n) \leq S(f, P_n, T_n) \leq U(f, P_n),$$

we see that

$$L(f, P_n) - \int_0^1 f \, dx \leq S(f, P_n, T_n) - \int_0^1 f \, dx \leq U(f, P_n) - \int_0^1 f \, dx,$$

and so

$$S(f, P_n, T_n) - \int_0^1 f \, dx \leq U(f, P_n) - L(f, P_n). \tag{3}$$

Similarly,

$$L(f, P_n) \leq \int_0^1 f \, dx \leq U(f, P_n),$$

and so

$$L(f, P_n) - S(f, P_n, T_n) \leq \int_0^1 f \, dx - S(f, P_n, T_n) \leq U(f, P_n) - S(f, P_n, T_n),$$

whence

$$\int_0^1 f \, dx - S(f, P_n, T_n) \leq U(f, P_n) - L(f, P_n). \tag{4}$$

Combining (3) and (4), we obtain the estimate

$$\left|\int_0^1 f\,dx - S(f,P_n,T_n)\right| \le U(f,P_n) - L(f,P_n). \tag{5}$$

Now, (2) implies that

$$U(f,P_n) - L(f,P_n) \le \frac{C}{n}, \tag{6}$$

as $\|P_n\| = 1/n$. It now follows from that (1) and (5)

$$\left|\int_0^1 f\,dx - \sum_{k=1}^n f\left(\frac{k}{n}\right)\frac{1}{n}\right| \le U(f,P_n) - L(f,P_n),$$

as was to be shown. □

Problem 4. Similarly as in Problem 3, we have the estimate

$$\omega_f(\delta) \le C\delta.$$

For a partition P of $[0,L]$, Homework 1, Problem 1(a) implies that

$$U(f,P) - L(f,P) \le C\|P\|L. \tag{7}$$

Fix $1/2 < \alpha < 1$. For each $n \in \mathbb{N}$, we define a partition $P_n' = \{x_0, \ldots, x_n\}$ of $[0, n^{1-\alpha}]$ and a set of points $T_n' = \{x_1, \ldots, x_n\}$ by setting $x_k = k/n^\alpha$. Then

$$S(f,P_n',T_n') = \frac{1}{n^\alpha}\sum_{k=1}^n f\left(\frac{k}{n^\alpha}\right). \tag{8}$$

Now, an argument analogous to the one given in Problem 3 yields the estimate

$$\left|\int_0^{n^{1-\alpha}} f\,dx - S(f,P_n',T_n')\right| \le U(f,P) - L(f,P),$$

whence (7) and (8) imply that

$$\left|\int_0^{n^{1-\alpha}} f\,dx - \frac{1}{n^\alpha}\sum_{k=1}^n f\left(\frac{k}{n^\alpha}\right)\right| \le C\cdot\frac{1}{n^\alpha}\cdot n^{1-\alpha} = C\cdot n^{1-2\alpha}.$$

Since $1/2 < \alpha < 1$, we see that $-1 < 1-2\alpha < 0$, whence

$$\lim_{n\to\infty}\left|\int_0^{n^{1-\alpha}} f\,dx - \frac{1}{n^\alpha}\sum_{k=1}^n f\left(\frac{k}{n^\alpha}\right)\right| \le \lim_{n\to\infty} Cn^{1-2\alpha} = 0.$$

As $1/2 < \alpha < 1$ also implies that $n^{1-\alpha} \to \infty$ as $n \to \infty$, the desired result now follows. □

Problem 5. `A pre-proof remark: as per your request, the proof below no longer depends on integration by parts.`

Let $f(x) = \cos(x^2)$ and, for each $n \in \mathbb{N}$, $x_n = \sqrt{(n-1/2)\pi}$. For each $a > 0$, the function f is continuous on the interval $[0,a]$, whence $f \in \mathscr{R}([0,a])$. This, in particular, implies that

$$b_n = \left|\int_{x_n}^{x_{n+1}} \cos(x^2)\,dx\right|$$

is a real number for each $n \in \mathbb{N}$. Note by periodicity of the cosine function that

$$b_n = \int_{x_n}^{x_{n+1}} |\cos(x^2)|\, dx.$$

Observe also that

$$\int_{x_n}^{x_{n+1}} \cos(x^2)\, dx = (-1)^n b_n \tag{9}$$

for all $n \in \mathbb{N}$, whence by the linearity of the integral we have the identity

$$\lim_{a\to\infty} \int_0^a \cos(x^2)\, dx = \int_0^{\sqrt{\pi/2}} \cos(x^2)\, dx + \sum_{n=1}^{\infty} (-1)^n b_n. \tag{10}$$

We claim that (10) is finite, which is the desired result. To this end, we note first that $\cos(x^2) \in \mathscr{R}([0, x_1])$, and so

$$\int_0^{x_1} \cos(x^2)\, dx < \infty.$$

It therefore suffices to show that

$$\sum_{n=1}^{\infty} (-1)^n b_n \tag{11}$$

is finite.

To this end, we recall the following result:

Lemma 5.1 (Leibniz alternating-series test)**.** *Let $(c_n)_{n=1}^{\infty}$ be a sequence of real numbers. If $(c_n)_{n=1}^{\infty}$ is a decreasing sequence of nonnegative numbers converging to zero, then*

$$\sum_{n=1}^{\infty} (-1)^n c_n$$

converges to a finite number.

Proof. Summation by parts. (You must have learned this in Analysis I, yes?) □

It thus suffices to check the hypothesis for $(b_n)_{n=1}^{\infty}$. By definition, $(b_n)_{n=1}^{\infty}$ is a sequence of nonnegative numbers. Since $\int_p^q g\, dx \le (\max_{a \le x \le b} |f|)\,(q-p)$ for each interval $[p, q]$ and every $g \in \mathscr{R}([p, q])$, we see that

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \int_{x_n}^{x_{n+1}} |\cos(x^2)|\, dx \\ &\le \lim_{n\to\infty} 1 \cdot \sqrt{\pi} \left(\sqrt{n + \frac{1}{2}} - \sqrt{n - \frac{1}{2}} \right) \\ &= 0. \end{aligned}$$

It remains to show that $(b_n)_{n=1}^{\infty}$ is a decreasing sequence. To this end, we shall make use of the following lemma:

Lemma 5.2 (Integration by substitution)**.** *Suppose that $g \in \mathscr{R}([p, q])$, $\varphi : [p', q'] \to [p, q]$ is an invertible differentiable map such that $\varphi(p') = p$ and $\varphi(q') = q$, and $(g \circ \varphi)\varphi' \in \mathscr{R}([p', q'])$. Then the formula*

$$\int_p^q g(x)\, dx = \int_{p'}^{q'} g(\varphi(x))\varphi'(x)\, dx$$

holds.

Proof of lemma. Let $F(t) = \int_p^t g(x)\,dx$. By the fundamental theorem of calculus,

$$\int_p^q g(x)\,dx = F(q) - F(p).$$

Now, $(F \circ \varphi)' = (F' \circ \varphi)\varphi'$ by the chain rule. It then follows from the fundamental theorem of calculus that

$$\int_{p'}^{q'} g(\varphi(x))\varphi'(x)\,dx = (F \circ \varphi)(q') - (F \circ \varphi)(p').$$

Since

$$(F \circ \varphi)(q') - (F \circ \varphi)(p') = F(\varphi(q')) - F(\varphi(p')) = F(q) - F(p),$$

the desired result follows. ☐

Fix $n \in \mathbb{N}$. Applying integration by substitution with the map $\varphi_1(t) = \sqrt{t}$, we see that

$$\begin{aligned} b_n &= \int_{\sqrt{(n-1/2)\pi}}^{\sqrt{(n+1/2)\pi}} |\cos(x^2)|\,dx \\ &= \int_{(n-1/2)\pi}^{(n+1/2)\pi} \frac{|\cos t|}{\sqrt{t}}\,dt. \end{aligned}$$

We invoke integration by substitution once more with the map $\varphi_2(s) = s + (n-1)\pi$ to obtain the identity

$$\begin{aligned} b_n &= \int_{(n-1/2)\pi}^{(n+1/2)\pi} \frac{|\cos t|}{\sqrt{t}}\,dt. \\ &= \int_{\pi/2}^{3\pi/2} \frac{|\cos(s + (n-1)\pi)|}{\sqrt{s + (n-1)\pi}}\,dt. \end{aligned}$$

By periodicity of cosine, we see that

$$b_n = \int_{\pi/2}^{3\pi/2} \frac{|\cos s|}{\sqrt{s + (n-1)\pi}}\,dt.$$

Since

$$\frac{|\cos s|}{\sqrt{s + (n-1)\pi}} \geq \frac{|\cos s|}{\sqrt{s + n\pi}}$$

for all $s \in [\pi/2, 3\pi/2]$, it follows that $b_n \geq b_{n+1}$.

We have thus shown that $(b_n)_{n=1}^{\infty}$ is a decreasing sequence of nonnegative numbers converging to zero. The Leibniz alternating-series test implies that (11) converges to a finite number. It now follows that $\cos(x^2)$ is improper Riemann-integrable on $[0, \infty)$, as was to be shown. ☐