Challenge Problem Set 6, Math 292 Spring 2012

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1 Introduction

This challenge problem set is about using the *Ritz method* to approximately compute eigenvalues for ordinary differential equations.

Consider the operator L acting on continuously differentiable functions $y(x)$ on $[0,1]$ defined by

$$Ly(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\big((1+x)y'(x)\big) + (2-x)y(x)\ .$$

As a consequence of th Strum Oscillation Theorems, here is an infinite sequence of numbers λ_n, $n = 1, 2, 3\dots$ such that there exists a continuously diffferentiabe function y_n satisfying the boundary conditions

$$y_n(0) = y_n(1) = 0$$

and such that

$$Ly_n = \lambda_n y_n\ .$$

For no other values of λ, besides those in the sequence, is there a non-zero solution of $Ly = y$ for y with $y(0) = y(1) = 0$. The numbers in the sequence $\{\lambda_n\}$ are the *eigenvalues* of the operator L, subject to the boundary conditions $y(0) = y(1) = 0$.

We suppose they are ordered so that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots\ .$$

Then, as explained in class, with K denoting the set of continuously differentiable functions that are zero at $x = 0$ and $x = 1$,

$$\lambda_1 = \text{g.l.b.}_{y\in K}\left\{ I[y]\ :\ \int_0^1 y^2(x)\mathrm{d}x\ \right\}$$

where

$$I[y] := \int_0^1 [(1+x)(y'(x))^2 + (2-x)(y(x))^2]\mathrm{d}x\ .$$

The *Ritz method* for computing this eigenvalue, and the corresponding eigenfunction $y_1(x)$, is to solve a sequence of constrained minimization problems in $\mathbb{R}^N$ for larger and larger values of N; i.e., a sequence of Lagrange multiplier problems.

For each positive integer N, define K_N to be the set of functions of the form

$$y(x) = \sum_{j=1}^{N} a_j x^j (1-x) \ , \qquad (a_1, \ldots, a_N) \in \mathbb{R}^N \ .$$

Notice that for each N, $K_N \subset K$. It follows that

$$\text{g.l.b.}_{y \in K_N} \left\{ I[y] \ : \ \int_0^1 y^2(x)\mathrm{d}x = 1 \right\} \geq \text{g.l.b.}_{y \in K} \left\{ I[y] \ : \ \int_0^1 y^2(x)\mathrm{d}x = 1 \right\} \ .$$

As explained in class, because of the Wierstrass theorem on polynomial approximation, it is the case that

$$\lim_{N\to\infty} \left(\text{g.l.b.}_{y \in K_N} \left\{ I[y] \ : \ \int_0^1 y^2(x)\mathrm{d}x = 1 \right\} \right) = \text{g.l.b.}_{y \in K} \left\{ I[y] \ : \ \int_0^1 y^2(x)\mathrm{d}x = 1 \right\} \ .$$

For any fixed N, the problem of computing

$$\text{g.l.b.}_{y \in K_N} \left\{ I[y] \ : \ \int_0^1 y^2(x)\mathrm{d}x = 1 \right\} \tag{1.1}$$

is simply a Lagrange multipliers problem in $\mathbb{R}^N$. As shown in class, one can hope to get good results for small values of N, say, $N = 2$, which is what we shall use here.

Taking $N = 2$, we avoid subscripts by writing

$$y(x) = ax(1-x) + bx^2(1-x) \tag{1.2}$$

for the general element of K_2.

Problem 1. Define a function $F(a,b)$ by $F(a,b) = I[y]$ with y given in terms of a, b by (1.2). Likewise, define a function $G(a,b)$ by $G(a,b) = \int_0^1 (y(x))^2\mathrm{d}x$. Compute explicit formulas for $F(a,b)$ and $G(a,b)$. Then the problem of computing the greatest lower bound in (1.1) for $N = 2$ is the problem of miniimizing $F(a,b)$ subject to the constraint $G(a,b) = 1$.

Problem 2. Notice that F and G are both quadratic functions on $\mathbb{R}^2$, meaning that there are 2×2 *symmetric* matrices A and B so that for all $(a,b) \in \mathbb{R}^2$, we have

$$F(a,b) = (a,b) \cdot A(a,b) \qquad \text{and} \qquad G(a,b) = (a,b) \cdot B(a,b) \ .$$

Find the matrices A and B. Show also that Lagrange's equation

$$\nabla F(a,b) = \lambda \nabla G(a,b)$$

is equivalent to the matrix equation

$$A\mathbf{a} = \lambda B\mathbf{a} \ ,$$

where $\mathbf{a} = (a,b)\ in\mathbb{R}^2$, which in turn is equivalent to

$$[A - \lambda B]\mathbf{a} = 0 \ .$$

Show that the Lagrange multiplier λ in the minimization problem must be one of the two roots of the quadratic polynomial

$$p_2(\lambda) := \det([A - \lambda B]) \ .$$

Compute the polynomial, and the two roots.

Problem 3. Show that subject to the constraint $G(a,b) = 1$, $F(a,b)$ has both a maximum and a minimum. Let λ be either of the roots of p_2 found above. Let (a_λ, b_λ) be any solution of the Lagrange system

$$\begin{aligned} [A - \lambda B](a,b) &= 0 \\ G(a,b) = 1 \ , \end{aligned}$$

Then taking the dot product of (a_λ, b_λ) with $[A - \lambda B](a_\lambda, b_\lambda)$ we get

$$(a_\lambda, b_\lambda) \cdot [A - \lambda B](a_\lambda, b_\lambda) = 0$$

since the vector on the right is zero by the first equation in the system above. Show that

$$f(a_\lambda, b_\lambda) = \lambda \ ,$$

and thus that the smaller of the two roots of $p_2(\lambda)$ is the minimum value of $F(a,b)$ subject to the constraint $G(a,b) = 1$. Compute the approximate value of λ_1 given by the Ritz method for $N = 2$. Call this number $\lambda_1^{(2)}$.

Problem 4. Find the values of a and b that minimize $F(a,b)$ subject to the constraint $G(a,b) = 1$, and define

$$u_2(x) := ax(1-x) + bx^2(1-x)$$

for these values of a and b. Compute and plot

$$Lu_2(x) - \lambda_1^{(2)} u_2(x) \ .$$

If we had the exact eigenvalue and eigenfunction, this would be zero. How did we do with $N = 2$?

Problem 5. There is also miniization problem that gives the second eigenvalue λ_2. Let y_1 denote the first eigenfunction, i.e., a non-zero function in K such that $Ly_1 = \lambda_1 y_1$, it is the case that

$$\lambda_2 = \text{g.l.b.}_{y \in K} \left\{ I[y] \ : \ \int_0^1 y^2(x)\mathrm{d}x = 1 \quad \text{and} \quad \int_0^1 y(x)y_1(x)\mathrm{d}x = 0 \right\} \ .$$

That is, now there are two constraints.

If we try an approximation at $N = 2$, the two constraint equations do not leave much leeway: There are only two solutions (a,b) of the system of constraint equations, and both give the same value for $F(a,b)$. Of course, one needs to know y_1, but let us take the approximation to it that we found in the last problem and use that. Solve the system system of equations

$$G(a,b) = 1 \qquad \int_0^1 y(x)u_2(x)\mathrm{d}x = 0$$

where $y(x) = a(x(1-x) + bx^2(1-x)$ and u_2 is the second order approximate eigenfunction found above. Use this solution to compute an approximate value for λ_2.

Concluding remarks: The methods used in this challenge problem set become very effective when used with software like Maple or MATLAB that allows one to easily use higher values of N. Also, part of the "art" of the method is using a good set of approximating functions; the sequence $\{\phi_n(x)\}$ with $\phi_n(x) = x^n(1-x)$ is not always the best choice, but it is often quite good.