Challenge Problem Set 3, Math 292 Spring 2012

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1 Introduction

This challenge problem set is about iteration and the contraction mapping theorem. In the exercises that follow, let $a > 0$, and define M to be the set of continuous real valued functions $x(t)$ defined on $-a \leq t \leq a$ with

$$x(0) = 1 \qquad \text{and} \qquad \max_{-a\leq s\leq a}\{|x(s)|\} \leq 2\ . \tag{1.1}$$

We equip M with the following metric: For any two functions x and y in M,

$$d(x,y) = \max_{-a\leq s\leq a}\{|x(s) - y(s)|\}\ .$$

1: Define a function Φ on M that transforms $x \in M$ into a new function $\Phi(x)$ given by

$$[\Phi(x)](t) = 1 + \int_0^t v(x(s))\mathrm{d}s \tag{1.2}$$

where

$$v(x) = -x\ . \tag{1.3}$$

For example, if $x(t) = 1 + t^2$,

$$[\Phi(x)](t) = 1 - \int_0^t (1+s^2)\mathrm{d}s = 1 - t - t^3/3\ .$$

(a) Define a sequence of functions $\{x^{(n)}(t)\}$ iteratively by

$$x^{(0)}(t) = 1 \qquad \text{and} \qquad x^{(n)}(t) = [\Phi(x^{(n-1)})](t) \quad \text{for all} \quad n \geq 1\ . \tag{1.4}$$

Compute explicit formulas for $x^{(1)}(t)$, $x^{(2)}(t)$, $x^{(3)}(t)$ and $x^{(4)}(t)$.

(b) Show that if $a \leq 1/2$, then $\Phi(x) \in M$ whenever $x \in M$. (Note that for any $x \in M$, $\Phi(x)$ satisfies $[\Phi(x)](0) = 1$ and $\Phi(x)$ is continuous – even differentiable. None of this depends on what a is. But if a is too large, $\Phi(x)$ may or may not satisfy the second condition in (1.1).)

(c) Show that if $a \leq 1/2$, then for all $x, y \in M$,

$$d(\Phi(x), \Phi(y)) \leq \frac{1}{2} d(x, y) \ .$$

(d) Show that $x_*(t) := e^{-t}$ is a fixed point of Φ; i.e., $[\Phi(x_*)](t) = x_*(t)$.

(e) Compute

$$d(x^{(0)}, x_*) \ , \qquad d(x^{(1)}, x_*) \ , \qquad d(x^{(2)}, x_*) \ , \qquad d(x^{(3)}, x_*) \ , \qquad \text{and} \qquad d(x^{(4)}, x_*) \ .$$

(f) Compute the fourth order Taylor approximant about $t = 0$ to x_*, and compare it with $x^{(4)}$.

(g) Show that x_* is the solution to

$$x'(t) = v(x(t) \qquad \text{with} \qquad x(0) = 1 \ .$$

2: This time, let

$$v(x) = x^2 \ .$$

Then, with this choice of v, define Φ by (1.2).

(a) Define a sequence of functions $\{x^{(n)}(t)\}$ iteratively by

$$x^{(0)}(t) = 1 \qquad \text{and} \qquad x^{(n)}(t) = [\Phi(x^{(n-1)})](t) \quad \text{for all} \quad n \geq 1 \ . \tag{1.5}$$

Compute explicit formulas for $x^{(1)}(t)$ and $x^{(2)}(t)$, $x^{(3)}(t)$.

(b) Find a value of $a > 0$ such that $\Phi(x) \in M$ whenever $x \in M$.

(c) Find a value of $a > 0$ such that for all $x, y \in M$,

$$d(\Phi(x), \Phi(y)) \leq \frac{1}{2} d(x, y) \ .$$

(d) Show that $x_*(t) := (1 - t)^{-1}$ is a fixed point of Φ; i.e., $[\Phi(x_*)](t) = x_*(t)$.

(e) Compute

$$d(x^{(0)}, x_*) \ , \qquad d(x^{(1)}, x_*) \ , \qquad \text{and} \qquad d(x^{(2)}, x_*) \ .$$

(f) Compute the third order Taylor approximant about $t = 0$ to x_*, and compare it with $x^{(3)}$.

(g) Show that x_* is the solution to

$$x'(t) = v(x(t) \qquad \text{with} \qquad x(0) = 1 \ .$$