SUGGESTED SOLUTIONS FOR PROBLEM SET 11

SPRING 2011, MATH 312:01

Exercise 1. We look at a variation of the function in the last problem of the preceeding set: n will be a natural number and we will be interested in the values as n approaches ∞. Let $f_n(x)$ be defined on $-\pi \leq x \leq \pi$ as follows:

$$f_n(x) = \begin{cases} 0 & \text{if } |x| > 1/n; \\ |n + n^2 x| & \text{if } |x| \leq 1/n. \end{cases}$$

(The earlier problem correspond to the case $n = 1$; the picture in this case is similar the picture in this case is similar except for a change of scale.) Note that once again each f_n is continuous, and in fact is piecewise linear. Compute the Fourier coefficients $\{a_0, a_j, b_j\}_{j\geq 1}$ and determine what happens to each as n approaches ∞.

(To make your life simpler, again use the fact that this function is even.)

Proof. Fix $n \in \mathbb{N}$, and observe that

$$a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f_n(x)\, dx = \frac{1}{\sqrt{2\pi}}.$$

and that, for $m \neq 0$,

$$\begin{aligned} a_m &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f_n(x) \cos(x)\, dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{1/n} (n - n^2 x) \cos(mx)\, dx \\ &= \frac{2}{\sqrt{\pi}} \left[\frac{n \sin(mx)}{m} - \frac{n^2 x \sin(mx)}{m} - \frac{n^2 \cos(mx)}{m^2} \right]_0^{1/n} \\ &= \frac{2}{\sqrt{\pi}} \left[\frac{n}{m} \sin\left(\frac{m}{n}\right) - \frac{n}{m} \sin\left(\frac{m}{n}\right) - \frac{n^2}{m^2} \cos\left(\frac{m}{n}\right) + \frac{n^2}{m^2} \right] \\ &= \frac{2}{\sqrt{\pi}} \cdot \frac{n^2}{m^2} \cdot \left[1 - \cos\left(\frac{m}{n}\right) \right]. \end{aligned}$$

We can apply L'Hôopital's rule to the real-variable counterpart of a_m to conclude that

$$\begin{aligned}
\lim_{n\to\infty} a_m &= \lim_{x\to\infty} \frac{2}{\sqrt{\pi}} \cdot \frac{x^2}{m^2} \cdot \left[1 - \cos\left(\frac{m}{x}\right)\right] \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{\pi}} \cdot \frac{-(m/x^2)\sin(m/x)}{-2(m^2/x^2)} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2m} \cdot x\sin\left(\frac{m}{x}\right) \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2m} \cdot \frac{-(m/x^2)\cos(m/x)}{-(1/x^2)} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2m} \cdot m \cdot \cos\left(\frac{m}{x}\right) \\
&= \frac{1}{\sqrt{\pi}}.
\end{aligned}$$

Finally, each b_n is 0, for f_n is even. □

Exercise 2. A little more about the Lipschitz condition, as mentioned in class. Let $f(x)$ satisfy a Lipschitz condition of order $\alpha > 1$ on $-\pi \le x \le \pi$. That is, suppose there is a constant C such that $|f(x) - f(z)| \le C|x - z|^\alpha$ for all x and z in the interval. Show that f is necessarily constant on the interval.

(Hint: show that f' exists at each point, and compute it, using the definition of derivative!)

Proof. Fix $\alpha > 1$, and suppose that $f : [-\pi, \pi] \to \mathbb{R}$ satisfies

$$|f(x) - f(z)| \le C|x - z|^\alpha$$

for all $x, z \in [-\pi, \pi]$, where C is a fixed constant independent of x and z. We note that

$$\left|\frac{f(x) - f(z)}{x - z}\right| \le C|x - z|^{\alpha - 1}$$

for all $x, z \in [-\pi, \pi]$, whence tending $z \to x$ yields $|f'(x)| \le 0$. It follows that the derivative of f is zero everywhere, and so the function is constant. □

Exercise 3. Let $f(x) = x$ on the interval $-\pi \le x \le \pi$. Compute the Fourier coefficients. On Monday we will see that this Fourier series must converge to f everywhere except possibly at the endpoints. (Note that f does not have values which match up at the endpoints, so is not continuous there in the periodic sense, let alone Lipschitz or differentiable.) What does happen to the Fourier series at the endpoints?

(Note: again you can simplify the computation because this function is odd.)

Proof. Since the function is odd, the Fourier series of f is a sine series, and so $a_n = 0$ for all $n \geq 0$. The sine coefficients are computed as follows:

$$\begin{aligned} b_n &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x \sin(nx)\, dx \\ &= \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} x \sin(nx)\, dx \\ &= \frac{2}{\sqrt{\pi}} \left[-\frac{x}{n} \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_0^{\pi} \\ &= \frac{2}{\sqrt{\pi}} \left(-\frac{\pi}{n} \cos(n\pi) \right) \\ &= \frac{(-1)^n 2\sqrt{\pi}}{n}. \end{aligned}$$

It follows that

$$f \sim \sum_{n=1}^{\infty} \frac{(-1)^n 2}{n} \sin(nx).$$

At the endpoints, $\sin(nx) = 0$ for all n, whence the corresponding Fourier series is zero. Evidently, zero is neither π nor $-\pi$. □

Exercise 4. Let f be the function defined on $[-\pi, \pi]$ as follows:

$$f(x) = \begin{cases} -1 & \text{for } -\pi \leq x \leq 0; \\ 1 & \text{for } 0 < x \leq \pi. \end{cases}$$

Compute the Fourier coefficients for f. This function has a jump discontinuity at 0, and at π (equivalently at $-\pi$). Again, on Monday we will see that this Fourier series must converge to f everywhere except possibly at 0 and π (equivalently at $-\pi$). What does happen to the Fourier series at the endpoints? What does convergence at $x = \pi/2$ tell you about the sum of the corresponding numerical series at that point?

(Again, note that f is odd.)

Proof. Since the function is odd, the Fourier series of f is a sine series, and so $a_n = 0$ for all $n \geq 0$. The sine coefficients are computed as follows:

$$\begin{aligned} b_n &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \\ &= \frac{2}{\sqrt{\pi}} \int_{0}^{\pi} \sin(nx)\, dx \\ &= \frac{2}{\sqrt{\pi}} \left[-\frac{1}{n} \cos(nx) \right]_0^{\pi} \\ &= \frac{2}{\sqrt{\pi}} \left[\frac{1}{n} - \frac{1}{n} \cos(n\pi) \right] \\ &= \frac{2[1 - (-1)^n)]}{n\sqrt{\pi}} \end{aligned}$$

It follows that

$$f \sim \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n)]}{n\pi} \sin(nx) = \sum_{n=1}^{\infty} \frac{4}{\sqrt{\pi}(2n-1)} \sin((2n-1)x).$$

At the endpoints, $\sin(nx) = 0$ for all n, whence the corresponding Fourier series is zero. Evidently, zero is neither 1 nor -1. Since the Fourier series converges to f at $x = 1/2$, we conclude that

$$\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi}{2}\right) = 1.$$

(Why $2n - 1$? $1 - (-1)^n$ is zero if n is even, and 2 if n is odd.) □