HONORS CALCULUS I, FALL 2015 - QUIZ 1 SOLUTIONS

Problem 1. Find the inverse of

$$A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

Solution. We apply the Gauss–Jordan method on the augmented matrix $[A \mid I]$:

$$[A \mid I] = \begin{bmatrix} 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left(-\frac{4}{3}\texttt{row1} + \texttt{row2}\right) \to \begin{bmatrix} 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(-6\texttt{row2} + \texttt{row1}) \to \begin{bmatrix} 3 & 0 & 0 & 0 & 9 & -6 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left(-\frac{7}{6}\texttt{row3} + \texttt{row4}\right) \to \begin{bmatrix} 3 & 0 & 0 & 0 & 9 & -6 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & -\frac{7}{6} & 1 \end{bmatrix}$$

$$(-30\texttt{row4} + \texttt{row3}) \to \begin{bmatrix} 3 & 0 & 0 & 0 & 9 & -6 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & -\frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 36 & -30 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & -\frac{7}{6} & 1 \end{bmatrix}$$

$$\begin{matrix} (\text{divide by } 3) \\ (\text{divide by } \frac{1}{3}) \\ (\text{divide by } 6) \\ (\text{divide by } \frac{1}{6}) \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 6 & -5 \\ 0 & 0 & 0 & 1 & 0 & 0 & -7 & 6 \end{bmatrix}.$$

We conclude that

$$A^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}.$$

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Problem 2. Let

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Find a lower triangular matrix L and a diagonal matrix D such that

$$B = LDL^T.$$

Solution. B is a symmetric matrix, and so the usual LDU decomposition yields $U = L^T$. DU is obtained by applying elimination without row exchanges to B:

$$\begin{aligned} B &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \\ (-\texttt{row1} + \texttt{row2}) &\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ (-\texttt{row1} + \texttt{row3}) &\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \\ (-\texttt{row2} + \texttt{row3}) &\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Since

$$DU = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

we see that

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover,

$$L = U^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

It follows that

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

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