SUGGESTED SOLUTIONS FOR PROBLEM SET 11

FALL 2010, MATH 311:01

6.1.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Prove that

$$\sum_{n=1}^{\infty}(a_n - a_{n+1})$$

converges iff $(a_n)_{n=1}^{\infty}$ converges. If $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ converges, what is the sum?

Proof. ($\Rightarrow$) Suppose that $\sum a_n - a_{n+1}$ converges to M, so that

$$\sum_{n=1}^{\infty} a_{n+1} - a_n = -\sum_{n=1}^{\infty} a_n - a_{n+1} = -M.$$

Since

$$\sum_{n=1}^{N} a_{n+1} - a_n = a_{N+1} - a_1$$

for each $N \in \mathbb{N}$, we see that

$$\begin{aligned}
\lim_{N\to\infty} a_N &= \lim_{N\to\infty} a_{N+1} \\
&= a_1 - a_1 + \lim_{N\to\infty} a_{N+1} \\
&= a_1 + \lim_{N\to\infty} a_{N+1} - a_1 \\
&= a_1 + \lim_{N\to\infty} \sum_{n=1}^{N} a_{n+1} - a_n \\
&= a_1 + \sum_{n=1}^{\infty} a_{n+1} - a_n \\
&= a_1 - M.
\end{aligned}$$

It follows that the sequence converges to $a_1 - M$.

($\Leftarrow$) We suppose that a_n converges to L as n approaches infinity. Since

$$\sum_{n=1}^{N} a_n - a_{n+1} = a_1 - a_{N+1}$$

for each $N \in \mathbb{N}$, we see that

$$\sum_{n=1}^{\infty} a_n - a_{n+1} = \lim_{N\to\infty} \sum_{n=1}^{N} a_n - a_{n+1} = \lim_{N\to\infty} a_1 - a_{N+1} = a_1 - L.$$

It follows that the series converges to $a_1 - L$, as was to be shown. □

6.1.2. Let $\sum_{n=1}^{\infty} a_n$ converge. Let $(n_k)_{k=1}^{\infty}$ be a subsequence of the sequence of positive integers. For each k, define

$$b_k = a_{n_{k-1}+1} + \cdots + a_{n_k} \quad \text{where} \quad n_0 = 0.$$

Prove that $\sum_{k=1}^{\infty} b_k$ converges and that

$$\sum_{k=1}^{\infty} b_k = \sum_{n=1}^{\infty} a_n.$$

Proof. For each $k \in \mathbb{N}$, we set

$$S_k = \sum_{n=1}^{k} a_n \qquad \text{and} \qquad T_k = \sum_{n=1}^{k} b_n = \sum_{n=1}^{n_k} a_n.$$

Noting that $T_k = S_{n_k}$, we see that

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k = \lim_{k\to\infty} S_{n_k} = \lim_{k\to\infty} T_k = \sum_{n=1}^{\infty} b_n,$$

as was to be shown. □

6.1.3. Prove that $\sum_{n=1}^{\infty} 2^n r^n$ convergers if $|r| < 1/2$ and find the sum.

Proof. For each $N \in \mathbb{N}$, we have

$$\sum_{n=0}^{N} 2^n r^n = \sum_{n=1}^{N} (2r)^n = \frac{1-(2r)^{N+1}}{1-2r}.$$

Letting N tend to infinity, we see that

$$\begin{aligned}
\sum_{n=1}^{\infty} 2^n r^n &= \lim_{N\to\infty} \sum_{n=1}^{N} 2^n r^n \\
&= \lim_{N\to\infty} \frac{1-(2r)^{N+1}}{1-2r} \\
&= \frac{1}{1-2r} - \lim_{N\to\infty} \frac{(2r)^{N+1}}{1-2r}.
\end{aligned}$$

If $|r| < 1/2$, then

$$\lim_{N\to\infty} \frac{(2r)^{N+1}}{1-2r} = \frac{1}{1-2r} \lim_{N\to\infty} (2r)^{N+1} = \frac{1}{1-2r} \cdot 0 = 0,$$

hence

$$\sum_{n=0}^{\infty} 2^n r^n = \frac{1}{1-2r}.$$

It now follows that

$$\sum_{n=1}^{\infty} 2^n r^n = -1 + \sum_{n=0}^{\infty} 2^n r^n = -1 + \frac{1}{1-2r}.$$

Of course, the series converges. □

6.1.4. Prove that the series $\sum_{n=0}^{\infty} 3^{-n}$ converges and find the limit.

Proof. Applying 6.1.3, we see that

$$\sum_{n=0}^{\infty} 3^{-n} = \sum_{n=0}^{\infty} \left(2 \cdot \frac{1}{6}\right)^n = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{6}\right)^n = \frac{1}{1-2(1/6)} = \frac{3}{2}.$$

□

6.1.5. Determine whether the series $\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n})$ converges or diverges. Justify your conclusion.

Proof. Since

$$\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n}) = \lim_{N\to\infty} \sum_{n=1}^{N} \sqrt{n+1} - \sqrt{n} = \lim_{N\to\infty} \sqrt{N+1} - 1 = \infty,$$

the series diverges. □

6.3.18. Give an example of an infinite series for which Theorem 6.8 yields a conclusion, but Theorem 6.9 does not.

Proof. Let $a_n = 2^{n^2}$ for each $n \in \mathbb{N}$, and consider the series $\sum a_n$. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{(n+1)^2}}{2^n}\right| = |2^{2n+1}| > 1,$$

Theorem 6.8 implies that the series diverges, whereas

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty,$$

whence the hypothesis of Theorem 6.9 is not satisfied. □

6.3.19. Use Theorem 6.9 to determine the values of r for which $\sum_{n=0}^{\infty} nr^n$ converges.

Proof. Observe that

$$\lim_{n\to\infty} \left|\frac{(n+1)r^{n+1}}{nr^n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| |r| = |r|.$$

The above quantity is less than 1 in case $|r| < 1$. □

6.3.20. Prove that $(nx^n)_{n=1}^{\infty}$ converges to zero if $|x| < 1$.

Proof. Suppose $|x| < 1$, and find any $M > 0$ such that $|x|^M < 1 - |x|$. Since

$$\lim_{n\to\infty} \sqrt[n]{n} = 1,$$

we may pick an integer N such that $n > N$ implies $\sqrt[n]{n} < 1 + |x|^{M-1}$. Then we have

$$\sqrt[n]{|nx^n|} = \sqrt[n]{n}|x| < |x| + |x|^M < 1$$

for each $n > N$, whence the root test implies that $\sum nx^n$ converges. It follows that $(nx^n)_{n=1}^{\infty}$ converges to zero. □

6.3.24a. Test the following series for convergence:

$$\sum_{n=1}^{\infty} n^p p^n, \qquad p > 0.$$

Proof. Note first that the series clearly diverges if $p = 1$. We observe that

$$\lim_{n\to\infty} \sqrt[n]{|n^p p^n|} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^p |p| = p \lim_{n\to\infty} \sqrt[n]{n}^p = p \cdot 1^p = p.$$

If $p < 1$, then we may fix $\varepsilon_1 > 0$ satisfying $p + \varepsilon_1 < 1$ and find an integer N_1 such that $n > N_1$ implies

$$\sqrt[n]{|n^p p^n|} < p + \varepsilon_1 < 1.$$

The convergence of the series then follows from the root test. If $p > 1$, then we may fix $\varepsilon_2 > 0$ satisfying $p - \varepsilon_2 > 1$ and find an integer N_2 such that $n > N_2$ implies

$$\sqrt[n]{|n^p p^n|} > p - \varepsilon_2 > 1.$$

Likewise, the divergence of the series follows from the root test. $\square$