SUGGESTED SOLUTIONS FOR PROBLEM SET 10

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5.5.19. Suppose $f \in \mathfrak{R}(x)$ on $[a,b]$ and $\frac{1}{f}$ is bounded on $[a,b]$. Prove that $\frac{1}{f} \in \mathfrak{R}(x)$ on $[a,b]$.

Pre-proof Remark. If f is continuous, then the proof is very simple. Since $\frac{1}{f}$ is bounded on $[a,b]$, the function $\phi(x) = 1/x$ is well-defined on $f([a,b])$, which is a compact set since f is continuous. ϕ is continuous on $\mathbb{R} \smallsetminus \{0\}$, whence by Theorem 5.11 $\phi \circ f = 1/f$ is Riemann-integrable on $[a,b]$.

...but f is not continuous! So we shall have to get our hands dirty a little bit.

Proof. Let $M > 0$ be any number such that $|1/f(x)| < M$ for all $x \in [a,b]$. Given any subinterval $[x_{i-1}, x_i] \subseteq [a,b]$ and points $x, y \in [x_{i-1}, x_i]$, we have the following estimate:

$$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| = \frac{|f(y) - f(x)|}{|f(x)||f(y)|} < \frac{M|f(y) - f(x)|}{|f(y)|} < M^2 |f(y) - f(x)| < M^2 \left[M_i(f) - m_i(f) \right].$$

Pick $\varepsilon > 0$, and find a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ such that

$$U(P,f) - L(P,f) < \frac{\varepsilon}{M^2}.$$

Then

$$\begin{aligned} U(P,1/f) - L(P,1/f) &= \sum_{i=1}^{n} [M_i(1/f) - m_i(1/f)](x_i - x_{i-1}) \\ &< M^2 \sum_{i=1}^{n} [M_i(f) - m_i(f)](x_i - x_{i-1}) \\ &= M^2 \left(U(P,f) - L(P,f) \right) \\ &< M^2 \cdot \frac{\varepsilon}{M^2} \\ &= \varepsilon, \end{aligned}$$

whence $1/f$ is Riemann-integrable on $[a,b]$. □

5.5.24. Suppose $g : [a,b] \to \mathbb{R}$ is bounded and continuous except at $x_1, \dots, x_n \in [a,b]$. Prove that $g \in \mathfrak{R}(x)$ on $[a,b]$.

Pre-proof Remark. Let us discuss the main ideas of the following (rather long) proof before indulging ourselves in the nitty-gritty technical details. We have seen that the key to establish the Riemann-integrability of a continuous function is that the function is *uniformly continuous* on closed intervals: that is, given a small enough subinterval (say of length smaller than δ), the total variation of the function is very small (within ε, for example).

When there are "a small enough number" of points of discontinuity, then some (but not all) subintervals will contain those "bad points," where the function is no longer uniformly continuous. Therefore, we cannot contain the total variation of the function in a small range on those subintervals. Since the function is assumed to be bounded, however, the total variation is still within some finite number. If we can bound the total length of the "bad" subintervals with a small enough number, then the product of the "variation on the y axis" and the "variation on the x axis"—which is what the "difference of the upper sum and the lower sum" is, essentially—will still be small.

Proof. Fix $\varepsilon > 0$, and find an $M > 0$ such that $|g(x)| < M$ for all $x \in [a, b]$. If $\varepsilon \geq 2M(b-a)$, then finding a partition P such that $U(P, f) - L(P, f) < \varepsilon$ is trivial. Therefore, we may well assume that $\varepsilon < 2M(b-a)$.

For each $1 \leq i \leq n$ we define an open interval $I_i = (x_i - \varepsilon/8Mn, x_i + \varepsilon/8Mn)$, and let

$$X = [a, b] \smallsetminus \bigcup_{i=1}^{n} I_i = [a, b] \cap \left(\bigcup_{i=1}^{n} I_i \right)^c.$$

$[a, b]$ is closed, and each I_i is open, hence X is closed. Furthermore, $X \subseteq [a, b]$, so that X is compact. Note also that X is nonempty, for the sum of the *length* of each I_i (namely, $(\varepsilon/4Mn) \cdot n = \varepsilon/4M$) is less than $b - a$.

g is continuous on the compact set X, hence g is uniformly continuous, and we may find $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon/2(b-a)$ for all $x, y \in X$. Now, let $P = \{p_0, \ldots, p_m\}$ be any partition of $[a, b]$ satisfying the following properties:

(1) The mesh of P is smaller than δ;
(2) P must contain both endpoints of each open interval I_i.

Take J to be the set of indices $\{1, \ldots, n\}$. We shall divide J into two parts as follows: if the subinterval $[p_{j-1}, p_j]$ is entirely contained in X, then $j \in J_g$; otherwise, $j \in J_b$.

We are now ready to make our estimate. We first observe that

$$U(P, f) - L(P, f) = \sum_{j=1}^{m} (M_j - m_j)(p_j - p_{j-1}) = \sum_{j \in J_g} (M_j - m_j)(p_j - p_{j-1}) + \sum_{k \in J_b} (M_k - m_k)(p_k - p_{k-1}).$$

Since $\mu(P) < \delta$, the "total variation" of f on each subinterval $[p_{j-1}, p_j] \subseteq X$ is within $\varepsilon/2(b-a)$. More specifically, condition (1) stipulates that f is uniformly continuous on the intevral $[p_{j-1}, p_j]$ of length less than δ, hence any $x, y \in [p_{j-1}, p_j]$ yields $|g(x) - g(y)| < \varepsilon$. Finally, $[p_{j-1}, p_j]$ is compact, whence we can find points $c_j, d_j \in [p_{j-1}, p_j]$ such that $g(c_j) = M_j$ and $g(d_j) = m_j$. We may thus conclude that

$$|M_j - m_j| = |g(c_j) - g(d_j)| < \varepsilon/2(b-a),$$

which immediately yields the inequality

$$\sum_{j \in J_g} (M_j - m_j)(p_j - p_{j-1}) < \sum_{j \in J_g} \frac{\varepsilon}{2(b-a)} (M_j - m_j) = \frac{\varepsilon}{2(b-a)} \sum_{j \in J_g} M_j - m_j.$$

$J_g \subseteq \{1, \ldots, n\}$, and each $M_j - m_j$ is postive, we have the inequality

$$\sum_{j \in J_g} M_j - m_j \leq \sum_{j=1}^{n} M_j - m_j = b - a,$$

and it follows that

$$\sum_{j \in J_g} (M_j - m_j)(p_j - p_{j-1}) < \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \frac{\varepsilon}{2}.$$

Now, condition (2) bounds the "total length" of the remaining subintervals. Indeed, (2) implies that each subinterval furnished by P is either in X or in $[a,b] \smallsetminus X$. It then follows that the union of the subintervals corresponding to the indices in J_b is precisely $[a,b] \smallsetminus X$, i.e.,

$$[a,b] \smallsetminus X = \bigcup_{j \in J_b} [p_{j-1}, p_j].$$

Since $[a,b] \smallsetminus X$ is the union of n intervals of length at most $\varepsilon/4Mn$, we have the following inequality:

$$\sum_{j \in J_b} p_j - p_{j-1} \leq n \cdot \frac{\varepsilon}{4Mn} = \frac{\varepsilon}{4M}.$$

In addition, g is bounded by M on $[a,b]$, so that $|g(x) - g(y)| < 2M$ for any $x, y \in [a,b]$. Therefore,

$$\sum_{j \in J_b} (M_i - m_i)(p_j - p_{j-1}) < 2M \sum_{j \in J_b} p_j - p_{j-1} \leq 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}.$$

It now follows that

$$U(P,f) - L(P,f) = \sum_{j \in J_g} (M_j - m_j)(p_j - p_{j-1}) + \sum_{k \in J_b} (M_k - m_k)(p_k - p_{k-1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whence $g \in \mathfrak{R}(x)$ on $[a,b]$, as was to be shown. □

Post-proof Remark. The same ideas can be used to establish the following generalized statement. We define a set $E \subseteq \mathbb{R}$ to be of ***measure zero*** if each $\varepsilon > 0$ admits a countable collection $\{(a_i, b_i)\}_{i \in \mathbb{N}}$ of open intervals such that $\sum |b_i - a_i| < \varepsilon$, and

$$E \subseteq \bigcup_{i \in \mathbb{N}} (a_i, b_i).$$

Then the following holds:

Theorem (Lebesgue's Criterion for Riemann Integrability—Sufficiency)**.** *Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let D be the set of points of discontinuities of f. If D is of measure zero, then f is Riemann-integrable on $[a,b]$.*

The proof is virtually identical to the one given above. We now have countable collection of "bad intervals," but we are still taking a finite partition, so there is no need to actually deal with infinity in the course of the proof.

Perhaps more surprising is the following

Theorem (Lebesgue's Criterion for Riemann Integrability—Necessity)**.** *Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let D be the set of points of discontinuities of f. If D is* not *of measure zero, then f is* not *Riemann-integrable on $[a,b]$.*

The proof is a bit long, so we omit it. Interested readers may consult pages 169-172 of T. M. Apostol's *Mathematical Analysis* (2e) for an elementary proof, or pages 203-207 of S. Abbott's *Understanding Analysis*, where the proof of Lebesgue's criterion is developed as a series of exercises.

Lebesgue's criterion provides a small glimpse into another theory of integration generally known as *Lebesgue integration*. In Lebesgue's theory, the notion of *measure* of sets is defined, and measures are used to define the integral of a function. We conclude this remark by quoting a famous description of the Lebesgue integral by Henri Lebesgue himself:

> I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken out all my money I order the bills and coins according to identical values and then I pay the several heaps one after another to the creditor. This is my integral.

5.5.27. Suppose f and g are integrable on $[a, b]$. Define $h(x) = \max\{f(x), g(x)\}$. prove that h is integrable on $[a, b]$.

Proof. Observe that

$$h(x) = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}$$

for all $x \in [a, b]$. The desired result now follows from Theorem 5.12 and Theorem 5.9(i). □

Remark. Analogously we have the following identity for any $a, b \in \mathbb{R}$:

$$\min\{a, b\} = \frac{a + b}{2} - \frac{|a - b|}{2}.$$

5.6.28. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f(t)\,dt = \int_x^1 f(t)\,dt$ for all $x \in [0, 1]$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

Proof. Define $F(x) = \int_0^x f(t)\,dt$ for each $x \in [0, 1]$. Since f is continuous, Theorem 5.14(ii) implies that F is differentiable and $F'(x) = f(x)$ for each $x \in [0, 1]$. Observe now that

$$F(1) = \int_0^1 f(t)\,dt = \int_1^1 f(t)\,dt = 0.$$

For each $x \in [0, 1]$, we have the equality

$$F(x) = \int_0^x f(t)\,dt = \int_x^1 f(t)\,dt = \int_0^1 f(t)\,dt - \int_0^x f(t)\,dt = F(1) - F(x) = -F(x),$$

whence $F(x) = 0$. Hence, F is a constant function on $[0, 1]$, and we have $F'(x) = f(x) = 0$ everywhere. □

5.6.29. Suppose f and g are continuous on $[a, b]$ and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$. Prove that there is $c \in [a, b]$ such that $f(c) = g(c)$.

Proof. Let $F(t) = \int_a^t f(x)\,dx$ and $G(t) = \int_a^t g(x)\,dx$ for each $t \in [a, b]$. f and g are continuous, hence Theorem 5.14(ii) implies that $F'(t) = f(t)$, and $G'(t) = g(t)$. The Cauchy mean-value theorem now furnishes a $c \in [a, b]$ such that

$$[F(b) - F(a)]G'(c) = [G(b) - G(a)]F'(c).$$

Since

$$F(b) - F(a) = F(b) = G(b) = G(b) - G(a)$$

it follows that

$$g(c) = G'(c) = F'(c) = f(c),$$

as was to be shown. □

5.6.30a. Find f' where f is defined on $[0,1]$ as indicated:

$$f(x) = \int_0^x \sqrt{t^2+1}\,dt.$$

Proof. $F(t) = \sqrt{t^2+1}$ is continuous on $[0,1]$, hence Theorem 5.14(ii) implies that

$$f'(x) = F(x) = \sqrt{x^2+1}.$$

□

5.6.30b. Find f' where f is defined on $[0,1]$ as indicated:

$$f(x) = \int_x^1 \cos\frac{1}{t+1}\,dt.$$

Proof. $F(t) = \cos\frac{1}{t+1}$ is continuous on $[0,1]$, hence Theorem 5.14(ii) implies that

$$f'(x) = F(x) = -\cos\frac{1}{x+1}.$$

□