SUGGESTED SOLUTIONS FOR PROBLEM SET 9

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5.1.1. Use theorem 5.2 to prove directly that the function $f(x) = x^3$ is integrable on $[0, 1]$.

Proof. For each $n \in \mathbb{N}$, we set

$$P_n = \left\{ \frac{k}{n} : 0 \le k \le n \right\},$$

which is a partition of $[0, 1]$. Then

$$U(P_n, f) = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \cdot \frac{1}{n} \quad \text{and} \quad L(P_n, f) = \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^3 \cdot \frac{1}{n},$$

so that

$$U(P_n, f) - L(P_n, f) = \frac{1}{n}.$$

Given any $\varepsilon > 0$, we may now pick a natural number $N > 1/\varepsilon$ to conclude that

$$U(P_N, f) - L(P_N, f) < \varepsilon.$$

It follows that f is Riemann-integrable on $[0, 1]$. □

5.1.2. Use Theorem 5.2 to prove directly that $f(x) = x$ is integrable on $[0, 1]$. Find the integral of f by finding a number A such that $L(P, f) \le A \le U(P, f)$ for all partitions of $[0, 1]$.

Proof. For each $n \in \mathbb{N}$, we set

$$P_n = \left\{ \frac{k}{n} : 0 \le k \le n \right\},$$

which is a partition of $[0, 1]$. Then

$$U(P_n, f) = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} \quad \text{and} \quad L(P_n, f) = \sum_{k=0}^{n-1} \frac{k}{n} \cdot \frac{1}{n},$$

so that

$$U(P_n, f) - L(P_n, f) = \frac{1}{n}.$$

Given any $\varepsilon > 0$, we may now pick a natural number $N > 1/\varepsilon$ to conclude that

$$U(P_N, f) - L(P_N, f) < \varepsilon.$$

It follows that f is Riemann-integrable on $[0, 1]$.

Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[0, 1]$. Since $m_i(f) = x_{i-1}$ and $M_i(f) = x_i$, we have

$$m_i(f) \le \frac{x_i + x_{i-1}}{2} \quad \text{and} \quad M_i(f) \ge \frac{x_i + x_{i-1}}{2}.$$

It then follows that

$$L(P,f)=\sum_{i=1}^{n} m_i(f)(x_i-x_{i-1}) \le \frac{1}{2}\sum_{i=1}^{n}(x_i+x_{i-1})(x_i-x_{i-1}) = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - x_{i-1}^2 = \frac{1}{2}$$

and

$$U(P,f)=\sum_{i=1}^{n} M_i(f)(x_i-x_{i-1}) \ge \frac{1}{2}\sum_{i=1}^{n}(x_i+x_{i-1})(x_i-x_{i-1}) = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - x_{i-1}^2 = \frac{1}{2},$$

so that

$$L(P,f) \le \frac{1}{2} U(P,f).$$

Since P was arbitrary, it follows that

$$\int_0^1 f(x)\,dx = \frac{1}{2}.$$

□

5.1.3. Define $f(x)=x$ if x is rational and $f(x)=0$ if x is irrational. Compute $\overline{\int_0^1} f\,dx$ and $\underline{\int_0^1} f\,dx$. Is f integrable on $[0,1]$? You may wish to look at the results of Exercise 2.

Proof. Note first that every nontrivial closed interval contains a rational and an irrational. Therefore, given any interval $[x_i, x_{i+1}]$, we see that

$$\sup\{f(x) : x \in [x_i, x_{i+1}]\} = x_{i+1} \qquad \text{and} \qquad \inf\{f(x) : x \in [x_i, x_{i+1}]\} = 0.$$

It then follows that

$$U(P,f)=\sum_{k=1}^{n} x_k(x_k - x_{k-1}) \qquad \text{and} L(P,f)=0,$$

where $P=\{x_0,\dots,x_n\}$ is a partition of $[0,1]$.

Recall from Exercise 5.1.2 that

$$\overline{\int_0^1} f\,dx = \inf_P U(P,f) = \frac{1}{2}.$$

Evidently,

$$\underline{\int_0^1} f\,dx = \sup_P L(P,f) = 0,$$

and we have $\overline{\int_0^1} f\,dx \ne \underline{\int_0^1} f\,dx$. It follows that f is not Riemann-integrable. □

5.1.4. A set $A \subseteq [0,1]$ is dense in $[0,1]$ iff every open interval that intersects $[0,1]$ contains a point of A. Suppose $f:[0,1]\to\mathbb{R}$ is integrable and $f(x)=0$ for all $x\in A$ with A dense in $[0,1]$. Show that $\int_0^1 f(x)\,dx = 0$.

Proof. Let $P=\{x_0,\dots,x_n\}$ be any partition of $[0,1]$. Then each (open) subinterval (x_{i-1},x_i) contains a point $t_i \in (x_{i-1},x_i)$ such that $f(t_i)=0$, so that $m_i \le 0$ and $M_i \ge 0$ for each i. Then

$$L(P,f)=\sum_{i=1}^{n} m_i(x_i-x_{i-1}) \le 0 \qquad \text{and} \qquad U(P,f)=\sum_{i=1}^{n} M_i(x_i-x_{i-1}) \ge 0.$$

Since P was arbitrary, we have

$$\overline{\int_0^1} f(x)\,dx \geq 0 \qquad \text{and} \qquad \underline{\int_0^1} f(x)\,dx \leq 0,$$

and the Riemann integrability of f on $[0,1]$ implies that

$$\int_0^1 f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \underline{\int_0^1} f(x)\,dx = 0,$$

as was to be shown. □

Remark. Looking ahead, we may obtain another proof using Riemann sums:

Fix $\varepsilon > 0$. f is integrable, hence we may find a $\delta > 0$ such that any partition P of $[0,1]$ with $\mu(P) < \delta$ satisfies $|S(P,f) - \int_0^1 f(x)\,dx| = 0$, regardless of how P is marked. Fix any such partition $P = \{x_0, \ldots, x_n\}$, and mark P as follows: each (open) subinterval (x_{i-1}, x_i) contains a point $t_i \in (x_{i-1}, x_i)$ such that $f(t_i) = 0$, for A is dense in $[0,1]$; we mark P with each such t_i. Then $S(P,f) = 0$, so that $|\int_0^1 f(x)\,dx| < \varepsilon$. Since ε was arbitrary, it follows that $|\int_0^1 f(x)\,dx| = 0$, or $\int_0^1 f(x)\,dx = 0$.

5.1.5. Define $f : [0,2] \to \mathbb{R}$ by $f(x) = 1$ for $0 \leq x \leq 1$ and $f(x) = 2$ for $1 < x \leq 2$. Show that $f \in \mathfrak{R}(x)$ on $[0,2]$ and compute the integral.

Proof. Fix $\varepsilon > 0$, and let $P_\varepsilon = \{0, 1 - \varepsilon/4, 1 + \varepsilon/4, 2\}$ be a partition of $[0,2]$. Then

$$U(P_\varepsilon, f) = 1\left(1 - \frac{\varepsilon}{4}\right) + 2\frac{\varepsilon}{2} + 2\left(1 - \frac{\varepsilon}{4}\right) = 2 \cdot \frac{\varepsilon}{2} + 3\left(1 - \frac{\varepsilon}{4}\right) = 3 + \frac{\varepsilon}{4}$$

and

$$L(P_\varepsilon, f) = 1\left(1 - \frac{\varepsilon}{4}\right) + \frac{\varepsilon}{2} + 2\left(1 - \frac{\varepsilon}{4}\right) = 1 \cdot \frac{\varepsilon}{2} + 3\left(1 - \frac{\varepsilon}{4}\right) = 3 - \frac{\varepsilon}{4},$$

so that $U(P,f) - L(P,f) = \varepsilon/2 < \varepsilon$. Therefore, f is Riemann-integrable on $[0,2]$. Observe now that each $\varepsilon > 0$ admits a partition P_ε, as defined above, such that

$$|U(P_\varepsilon, f) - 3| < \varepsilon \qquad \text{and} \qquad |L(P_\varepsilon, f) - 3| < \varepsilon.$$

We thus conclude that

$$\overline{\int_0^2} f(x)\,dx = 3 \qquad \text{and} \qquad \underline{\int_0^2} f(x)\,dx = 3,$$

which implies that

$$\int_0^2 f(x)\,dx = 3.$$

□

5.2.7. Suppose $g : [a,b] \to \mathbb{R}$ is continuous except at $x_0 \in (a,b)$ and bounded. Prove that $g \in \mathfrak{R}(x)$ on $[a,b]$. See Exercises 24 and 25 for generalizations of this result.

Proof. Fix $\varepsilon > 0$, and let

$$M = \sup\{f(x) : x \in [a,b]\} \qquad \text{and} \qquad m = \inf\{f(x) : x \in [a,b]|\};$$

furthermore, we set $L = M - m$. g is continuous on $[a, x_0 - \varepsilon/6L]$, hence g is Riemann-integrable on $[a, x_0 - \varepsilon/6L]$, and there exists a partition P_1 of $[a, x_0 - \varepsilon/6L]$ such that $U(P_1, f) - L(P_1, f) < \varepsilon/3$.

Likewise, g is continuous on $[x_0 + \varepsilon/6L, b]$, and we may find a partition P_2 of $[x_0 + \varepsilon/6L]$ such that $U(P_2, f) - L(P_2, f) < \varepsilon/3$.

We now set $P = P_1 \cup P_2$, and let

$$M' = \sup\{f(x) : x \in [x_0 - \varepsilon/6L, x_0 + \varepsilon/6L]\}$$

and

$$m' = \inf\{f(x) : x \in [x_0 - \varepsilon/6L, x_0 + \varepsilon/6L]\}.$$

Then

$$U(P, f) = U(P_1, f) + U(P_2, f) + M' \cdot \frac{\varepsilon}{3L}$$

and

$$L(P, f) = L(P_1, f) + U(P_2, f) + m' \cdot \frac{\varepsilon}{3L},$$

so that

$$\begin{aligned} U(P, f) - L(P, f) &= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] + (M' - m') \cdot \frac{\varepsilon}{3L} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + L \cdot \frac{\varepsilon}{3L} \\ &= \varepsilon. \end{aligned}$$

Note that we have used the inequality $M' - m' \leq M - m = L$. It thus follows that f is Riemann-integrable on $[a, b]$. □

5.2.9. Assume $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f\,dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

Proof. We suppose for a contradiction that $f(x_0) \neq 0$ for some $x_0 \in [a, b]$. We can then find a $\delta > 0$ such that $|f(x) - f(x_0)| < f(x_0)/2$, i.e., $f(x) > f(x_0)/2$, on $(x_0 - \delta, x_0 + \delta)$. Since $f(x) \geq 0$ for all $x \in [a, b]$, it follows that

$$U(P, f) > \frac{f(x_0)}{2} \cdot 2\delta = f(x_0)$$

for each partition P, whence the upper integral of f cannot be 0. This is evidently absurd, for the value of $\int_a^b f\,dx$ was assumed to be zero. We thus conclude that $f(x) = 0$ for all $x \in [a, b]$. □

5.3.15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is periodic and integrable on every closed interval. If p is the period of f, prove that for any $a \in \mathbb{R}$,

$$\int_0^p f\,dx = \int_a^{a+p} f\,dx.$$

Proof. For each partition $P = \{x_0, \ldots, x_n\}$ of $[0, p]$, we define a partition

$$P^{+a} = \{x_0 + a, \ldots, x_n + a\}$$

of $[a, a + p]$. Find $n \in \mathbb{Z}$ such that $0 \leq a + np < p$; the existence of such an n is guaranteed by the Archimedean property. Given this n, we set $a' = a + np$. Then

$$f(a' + x) = f(a + np + x) = f(a + x)$$

for any $x \in \mathbb{R}$.

Fix $\varepsilon > 0$. Since f is integrable on $[a', p]$, we may find a marked partition P_1 such that

$$\left| S(Q_1, f) - \int_{a'}^p f(x)\,dx \right| < \varepsilon$$

for any refinement Q of P_1. Then P_1^{+np} is a marked partition of $[a,(n+1)p]$, and

$$\begin{aligned} S(P_1,f) &= \sum f(t_i)(x_i - x_{i-1}) \\ &= \sum f(t_i+np)(x_i - x_{i-1}) \\ &= \sum f(t_i+np)\left[(x_i+np)-(x_{i-1}+np)\right] \\ &= S(P_1^{+np}, f). \end{aligned}$$

The same reasoning applies to any refinement Q_1 of P_1, so that $S(Q_1,f) = S(Q_1^{+np},f)$, and

$$\left| S(Q_1^{+np},f) - \int_{a'}^{p} f(x)\,dx \right| < \varepsilon.$$

Given any refinement Q of P_1^{+np}, the partition Q^{-np} of $[a',p]$ is a refinement of $\left(P_1^{+np}\right)^{+(-np)} = P_1$, and we have $\left(Q^{+(-np)}\right)^{+np} = Q$. Therefore, the above inequality holds for all refinements of P_1^{+np}, whence

$$\int_a^{(n+1)p} f(x)\,dx = \int_{a'}^{p} f(x)\,dx.$$

Likewise, f is integrable on $[0,a']$, so we may find a marked partition P_2 such that

$$\left| S(Q_2,f) - \int_0^{a'} f(x)\,dx \right| < \varepsilon$$

for any refinement Q_2 of P_1. By the same reasoning as above, every refinement Q of the partition $P_2^{+(n+1)p}$ of the interval $[(n+1)p, a'+(n+1)p] = [(n+1)p, a+p]$ satisfies the inequality

$$\left| S(Q,f) - \int_0^{a'} f(x)\,dx \right| < \varepsilon,$$

whence

$$\int_{(n+1)p}^{p+a} f(x)\,dx = \int_0^{a'} f(x)\,dx.$$

It now suffices to show that

$$\int_a^{(n+1)p} f(x)\,dx + \int_{(n+1)p}^{a+p} f(x)\,dx = \int_a^{a+p} f(x)\,dx$$

and

$$\int_0^{a'} f(x)\,dx + \int_{a'}^{p} f(x)\,dx = \int_0^{p} f(x)\,dx.$$

These equalities are direct consequences of the following

Lemma. *If $f : [b,c] \to \mathbb{R}$ is a bounded, Riemann-integrable function, then*

$$\int_b^d f(x)\,dx + \int_d^c f(x)\,dx = \int_b^c f(x)\,dx$$

for any $d \in (b,c)$; of course, it must also follow that f is Riemann-integrable on $[b,d]$ and $[d,c]$.

Proof of the lemma. Fix $\varepsilon > 0$, and pick a partition P of $[b, c]$ such that $U(P, f) - L(P, f) < \varepsilon$. Let Q be a refinement of Q given by $Q = P \cup \{d\}$, and set $Q_1 = Q \cap [b, d]$ and $Q_2 = Q \cap [d, c]$. Then

$$\varepsilon > U(Q, f) - L(Q, f) = [U(Q_1, f) - L(Q_1, f)] + [U(Q_2, f) - L(Q_2, f)].$$

Since $U(Q_i, f) - L(Q_i, f) \geq 0$ for each i, we have $U(Q_i, f) - L(Q_i, f) < \varepsilon$ for each i. Therefore, f is Riemann-integrable on $[b, d]$ and $[d, c]$.

Fix partitions P_1 and P_2 of $[b, d]$ and $[d, c]$, respectively, so that any refinements P_1' and P_2' of P_1 and P_2, respectively, satisfy

$$\left| S(P_1', f) - \int_b^d f\,dx \right| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \left| S(P_2', f) - \int_d^c f\,dx \right| < \frac{\varepsilon}{2}.$$

Let $\mathfrak{P} = P_1 \cup P_2$. Then $\mathfrak{P}$ is a partition of $[b, c]$. Furthermore, if $\mathfrak{Q}$ is any refinement of $\mathfrak{P}$, then $\mathfrak{Q}_1 = \mathfrak{Q} \cap [b, d]$ and $\mathfrak{Q}_2 = \mathfrak{Q} \cap [d, c]$ are refinements of P_1 and P_2, respectively. Therefore,

$$\begin{aligned}
\left| S(\mathfrak{Q}, f) - \left[\int_b^d f\,dx + \int_d^c f\,dx \right] \right| &= \left| S(\mathfrak{Q}_1, f) + S(\mathfrak{Q}_2, f) - \int_b^d f\,dx - \int_d^c f\,dx \right| \\
&\leq \left| S(\mathfrak{Q}_1, f) - \int_b^d f\,dx \right| + \left| S(\mathfrak{Q}_2, f) - \int_d^c f\,dx \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

whence

$$\int_b^c f\,dx = \int_b^d f\,dx + \int_d^c dx,$$

as was to be shown. □

The desired result now follows from the above lemma. □

Remark. Admittedly, the above proof is unnecessarily complicated. Here is a way to establish the above result, if we were allowed to freely wield the theorems in the later sections of Chapter 5.

Find $n \in \mathbb{Z}$ such that $a' = a - np$ and $0 \leq a' < p$. Since f is p-periodic, a repeated application of the change-of-variables theorem and the "algebra of integrable functions" theorem gives:

$$\begin{aligned}
\int_0^p f\,dx &= \int_0^{a'} f\,dx + \int_{a'}^p f\,dx = \int_{np}^{a'+np} f\,dx + \int_{a'+np}^{(n+1)p} f\,dx \\
&= \int_{np}^a f\,dx + \int_a^{(n+1)p} f\,dx = \int_{(n+1)p}^{p+a} f\,dx + \int_a^{(n+1)p} f\,dx \\
&= \int_a^{p+a} f\,dx.
\end{aligned}$$

5.3.16c. Use the Fundamental Theorem of Integral Calculus to compute the following:

$$\int_0^{\pi/2} x \sin x^2\,dx.$$

Proof. Let $f(x) = -\frac{1}{2}\cos x^2$, and recall that $f'(x) = x \sin x^2$. By the fundamental theorem of integral calculus,

$$\int_0^{\pi/2} x \sin x^2 \, dx = -\frac{1}{2}\cos\frac{\pi^2}{4} + \frac{1}{2}\cos 0 = \cos\frac{\pi^2}{4} + \frac{1}{2}.$$

□