SUGGESTED SOLUTIONS FOR PROBLEM SET 8

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4.2.12. Suppose $f : [a,b] \to [c,d]$, $g : [c,d] \to [p,q]$, and $h : [p,q] \to \mathbb{R}$, with f differentiable at $x_0 \in [a,b]$, g differentiable at $f(x_0)$, and h differentiable at $g(f(x_0))$. Prove that $h \circ (g \circ f)$ is differentiable at x_0, and find the derivative.

Proof. By the chain rule, $g \circ f$ is differentiable at $f(x_0)$, and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$. Applying the chain rule again, we see that $h \circ (g \circ f)$ is differentiable at $g(f(x_0))$, and

$$(h \circ (g \circ f))'(x_0) = h'(g(f(x_0)))(g \circ f)'(x_0) = h'(g(f(x_0)))g'(f(x_0))f'(x_0).$$

□

4.3.17. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(1 + x^2)$. Prove that f has a maximum value and find the point at which that maximum occurs.

Proof. We claim that f has a maximum at 0. Indeed, $1 \leq 1 + x^2$ for all $x \in \mathbb{R}$, whence it follows that

$$f(0) = 1 \geq \frac{1}{1 + x^2} = f(x)$$

for each $x \in \mathbb{R}$. □

Remark. To show that $f'(0) = 0$ without explicit computations, we note that each $x > 0$ admits a real number $c_x \in (-x, x)$ such that

$$f(c_x) = \frac{f(x) - f(-x)}{2x} = 0$$

via Rolle's theorem, and observe that

$$\bigcap_{x>0} (-x, x) = \{0\},$$

whence $f'(0) = 0$.

4.3.18. Prove that the equation $x^3 - 3x + b = 0$ has at most one root in the interval $[-1, 1]$.

Proof. Set $f(x) = x^3 - 3x + b$, and observe that $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$. Since $(x+1) < 0$ and $(x-1) > 0$ on $(-1,1)$, the derivative $f'(x)$ is strictly negative on $(-1,1)$. In fact, f is strictly decreasing on $[-1,1]$. Indeed, $f(-1) = f(x_0)$ for some $x_0 \in (-1,1)$, then Rolle's theorem furnishes $a \in (-1, x_0)$ such that $f'(d) = 0$, which is evidently absurd. Supposing that $f(1) = f(x_0)$ holds for some $x_0 \in (-1,1)$ leads to the same contradiction. It now suffices to observe that f is one-to-one on $[-1,1]$, whence it follows that at most one $c \in [-1,1]$ can satisfy $f(c) = 0$, as was to be shown. □

4.3.20. Suppose $f : [0,2] \to \mathbb{R}$ is differentiable, $f(0) = 0$, $f(1) = 2$, and $f(2) = 2$. Prove that

(1) there is c_1 such that $f'(c_1) = 0$
(2) there is c_2 such that $f'(c_2) = 2$, and
(3) there is c_3 such that $f'(c_3) = \frac{3}{2}$.

Proof. These are routine applications of the mean value theorem.

$$\frac{f(2)-f(1)}{2-1}=0,$$

furnishes $c_1 \in (1,2)$ such that $f'(c_1)=0$, and

$$\frac{f(1)-f(0)}{1-0}=2$$

yields $c_2 \in (0,1)$ such that $f'(c_2)=2$. (3) now follows from Darboux's theorem (Theorem 4.11), for $f'(c_1) < 3/2 < f'(c_2)$. □

4.3.23. Use the Mean-Value Theorem to prove that

$$\sqrt{1+h} < 1+\frac{1}{2}h \qquad \text{for} \qquad h>0.$$

Proof. Define $f:(0,\infty)\to\mathbb{R}$ by $f(x)=\sqrt{1+x}$. For each $h>0$, the mean-value theorem furnishes $c_h \in (0,h)$ such that

$$f'(c_h)=\frac{f(h)-f(0)}{h-0}=\frac{\sqrt{1+h}-1}{h}.$$

Recalling that $1<\sqrt{1+x}$ for all $x>0$, we have

$$f'(c_h)=\frac{1}{2\sqrt{1+c_h}}<\frac{1}{2},$$

which implies

$$\frac{\sqrt{1+h}-1}{h}<\frac{1}{2}.$$

Rearranging the above inequality yields

$$\sqrt{1+h}<1+\frac{1}{2}h,$$

which is the desired result. □

4.3.24. Generalize Exercise 23 as follows: If $0<p<1$ and $h>0$, then show that

$$(1+h)^p < 1+ph.$$

Proof. Define $f:(0,\infty)\to\mathbb{R}$ by $f(x)=(1+x)^p$. For each $h>0$, the mean-value theorem furnishes $c_h\in(0,h)$ such that

$$f'(c_h)=\frac{f(h)-f(0)}{h-0}=\frac{(1+h^p-1}{h}.$$

Recalling that $1>(1+x)^{p-1}$ for all $x>0$, we have

$$f'(c_h)=p(1+h)^{p-1}<p,$$

which implies

$$\frac{(1+h)^p-1}{h}<p.$$

Rearranging the above inequality yields

$$(1+h)^p<1+ph,$$

which is the desired result. □

4.3.28. Prove that the function $f(x) = 2x^3 + 3x^2 - 36x + 5$ is 1-1 on the interval $[-1, 1]$. Is f increasing or decreasing?

Proof. We first note that $f'(x) = 6x^2 + 6x - 36 = 6(x - 2)(x + 3)$. Since $x - 2 < 0$ and $x + 3 > 0$ on $(-3, 2)$, f' is strictly negative on $[-1, 1]$. Since f' is nonzero on $[-1, 1]$, f is 1-1 on $[-1, 1]$; furthermore, f is deacreasing on the interval. □

4.3.29. Prove that the function $f(x) = x^3 - 3x^2 + 17$ is not 1-1 on the interval $[-1, 1]$.

Proof. Since $f(1) = f(-1)$, the mean-value theorem furnishes a $c \in (-1, 1)$ such that $f'(c) = 0$. Fix any d between $f(-1)$ and $f(c)$. f is continuous, hence the intermediate value theorem furnishes a $d_1 \in (-1, c)$ such that $f(d_1) = d$. Since d is also in between $f(c)$ and $f(1)$, the intermediate value theorem furnishes another real number $d_2 \in (c, 1)$ such that $f(d_2) = d$. It follows that f is not one-to-one on $[-1, 1]$. □

4.4.33. Use L'Hospital's Rule to find the limit

$$\lim_{x\to 0} \frac{x^2 \sin x}{\sin x - x \cos x}.$$

Proof. We first note that $\lim_{x\to 0} x^2 \sin x = 0$ and $\lim_{x\to 0} \sin x - x \cos x = 0$. By L'Hospital's rule,

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2 \sin x}{\sin x - x \cos x} &= \lim_{x\to 0} \frac{x^2 \cos x + 2x \sin x}{x \sin x} \\
&= \lim_{x\to 0} \frac{x \cos x}{\sin x} + 2 \\
&= \lim_{x\to 0} \frac{\cos x - x \sin x}{\cos x} + 2 \\
&= 1 + 2 \\
&= 3.
\end{aligned}$$

□

4.4.35. Find an equation for the line tangent to the graph of f^{-1} at the point $(3, 1)$ if $f(x) = x^3 + 2x^2 - x + 1$.

Proof. We first note that $f(1) = 3$; we must thus compute $(f^{-1})'(3)$. Computing the derivative, we have $f'(x) = 3x^2 + 4x - 1$. Since f' is a concave-down parabola with zeroes at $(-2 - \sqrt{7})/3$ and $(-2 + \sqrt{7})/3$, $f'(x) > 0$ on some interval including 1, say, $[0.5, 3]$. Therefore, the inverse function theorem implies that $f^{-1}(3)$ exists and

$$(f^{-1})'(3) = (f^{-1})'(f(1)) = \frac{1}{f'(1)} = \frac{1}{6}.$$

Therefore, the line tangent to the graph of f^{-1} at the point $(3, 1)$ has the slope of $1/6$. Routine computations show that the equation of the line is

$$y = \frac{1}{6}x + \frac{1}{2}.$$

□