SUGGESTED SOLUTIONS FOR PROBLEM SET 7

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3.3.36. If $E_1, \dots, E_n$ are compact, prove that $E = \bigcup_{i=1}^n$ is compact.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in J\}$ be an open cover of E. For $1 \le i \le n$, the compact set E_i is a subset of E, thus covered by $\mathcal{U}$; we may thus extract a finite subcover $\{U_k^i : 1 \le k \le N_i\}$. Let

$$\mathcal{O} = \{U_k^i : 1 \le i \le n,\, 1 \le k \le N_i\},$$

which is a finite subcollection of $\mathcal{U}$. For each i, our collection $\mathcal{O}$ includes a cover of E_i, whence $\mathcal{O}$ covers E. Therefore, $\mathcal{O}$ is a finite subcover of $\mathcal{U}$ covering E, and we conclude that E is compact. □

3.3.37. Let $f : [a, b] \to \mathbb{R}$ have a limit at each $x \in [a, b]$. Prove that f is bounded.

Proof. For each $x_0 \in [a, b]$, we let

$$L_{x_0} = \lim_{x \to x_0} f(x).$$

Fix $\varepsilon > 0$. For each $x \in [a, b]$, we find $\delta_x > 0$ such that $y \in [a, b]$ and $|x - y| < \delta_x$ implies $|L_x - f(y)| < \varepsilon$. We set $U_x = (x - \delta_x, x + \delta_x)$ and $V_x = (L_x - \varepsilon, L_x + \varepsilon)$ so that $y \in U_x \cap [a, b]$ implies $f(y) \in V_x$.

Observe that the collection $\{U_x : x \in [a, b]\}$ is an open cover of the compact interval $[a, b]$, which admits a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$. Therefore,

$$[a, b] \subseteq \bigcup_{i=1}^n U_{x_i},$$

whence

$$f([a, b]) = \{f(x) : x \in [a, b]\} \subseteq \bigcup_{i=1}^n V_{x_i}$$

by the above construction. Each V_{x_i} is bounded, and a finite union of bounded sets is bounded, hence $V_{x_1} \cup \dots \cup V_{x_n}$ is bounded. Since any subset of a bounded set is bounded, $f([a, b])$ is bounded. It now follows that f is a bounded. □

Remark. A counterexample on $[0, 1]$, without the existence-of-limit condition:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational;} \\ 1/x & \text{if } x \text{ is irrational.} \end{cases}$$

3.4.41. Find an interval of length 1 that contains a root of the equation $xe^x = 1$.

Proof. Let $f(x) = xe^x$. We take for granted that f is continuous, whence we may apply the intermediate value theorem to f. Observe that $f(0) = 0$ and $f(1) = e$. Since $0 < 1 < e$, it follows from the intermediate value theorem that $f(x) = 1$ for some $x \in (0, 1)$, as was to be shown. □

Remark. We have taken for granted that $f(x) = xe^x$ is continuous in the above exercise, as we cannot establish the continuity of f with the tools provided in the course thus far. Indeed, we have yet to define what it means to *exponentiate* a real number by another real number. Herein we present a brief version of the standard construction of the exponential function e^x.

Given a sequence $(a_n)_{n=1}^{\infty}$ of real numbers, we define the *sum* of the sequence $(a_n)_{n=1}^{\infty}$ to be the limit of the sequence $(s_n)_{n=1}^{\infty}$, where

$$s_n = a_1 + a_2 + \cdots + a_n$$

for each $n \in \mathbb{N}$. The sum is denoted by

$$\sum_{n=1}^{\infty} a_n,$$

and is referred to as a *series*. If $(s_n)_{n=1}^{\infty}$ converges to a real number, we say that the series *converges*; otherwise, the series *diverges*. A sequence $(a_n)_{n=1}^{\infty}$ and a (real) variable x, determines a *(real) power series*

$$\sum_{n=1}^{\infty} a_n x^n,$$

which is a series for each $x \in \mathbb{R}$.

We now define the *exponential function* $\exp : \mathbb{R} \to (0, \infty)$ by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1}$$

By the ratio test, exp converges everywhere on $\mathbb{R}$. We set $\exp(1) = e$, and write e^x for $\exp(x)$. It follows from the definition that $e^0 = 1$, and $e^{x+y} = e^x e^y$ for all $x, y \in \mathbb{R}$.

Furthermore, given any $x \in \mathbb{R}$, we see that

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x,$$

where the last equality follows from (1):

$$\frac{e^h - 1}{h} = -\frac{1}{h} + \frac{1}{h} \sum_{n=0}^{\infty} \frac{h^n}{n!} = -\frac{1}{h} + \frac{1}{h} + \sum_{n=1}^{\infty} \frac{h^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{h^n}{(n+1)!} = 1 + \sum_{n=1}^{\infty} \frac{h^n}{(n+1)!}.$$

Therefore, exp is differentiable everywhere, and $\exp'(x) = \exp(x)$. It also follows that exp is continuous everywhere.

To define the exponentiation of any positive real number, we first define the *logarithmic function* $\log : (0, \infty) \to \mathbb{R}$ by

$$\log(x) = \int_1^x \frac{dt}{t}.$$

Since the integral of a continuous function is continuous, log is continuous on $(0, \infty)$. log is the *inverse function* of exp, viz., $\log(\exp(x)) = x$ for all $x \in \mathbb{R}$, and $\exp(\log(y)) = y$ for all $y \in (0, \infty)$. Now, given any real number $b > 0$, we define

$$b^x = e^{x \cdot \log(b)}$$

for all $x \in \mathbb{R}$. It is a composition of continuous functions, hence continuous. Note that

$$e^{x \cdot \log(b)} = \left(e^{\log(b)}\right)^x = (\exp(\log(b)))^x = b^x$$

in the usual rules of exponents, so that the definition agrees with older notions.

3.4.42. Find an interval of length 1 that contains a root of the equation $x^3 - 6x^2 + 2.826 = 0$.

Proof. For each $n \in \mathbb{R}$, the monomial function $f_n(x) = x^n$ is continuous, whence all polynomial functions are continuous. In particular, $p(x) = x^3 - 6x^2 + 2.826$ is continuous. Observe that $p(0) = 2.826$ and $p(1) = -2.174$. Since $-2.174 < 0 < 2.826$, it follows from the intermediate value theorem that $p(x) = 0$ for some $x \in (0, 1)$, as was to be shown. □

4.1.3. Use the definition to find the derivative of $f(x) = \sqrt{x}$, for $x > 0$. Is f differentiable at zero? Explain.

Proof. Fix $x > 0$. We observe that

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

We also note that

$$\lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h\to 0^+} \frac{\sqrt{h}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty,$$

whence f is not differentiable at 0. □

4.1.4. Use the definition to find the derivative of $g(x) = x^2$.

Proof. For each $x \in \mathbb{R}$, we have the following:

$$g'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} 2x + h = 2x.$$

□

4.1.6. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable at $x \in (a, b)$. Prove that

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists and equals $f'(x)$. Give an example of a function where this limit exists, but the funciton is not differentiable.

Proof. f is differentiable, hence

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Since the limit exists, both the left limit and the right limit are equal to the limit, viz.,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{(-h)\to 0^+} \frac{f(x-h) - f(x)}{-h} \\
&= \lim_{(-h)\to 0^+} \frac{f(x) - f(x-h)}{h}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{(-h)\to 0^-}\frac{f(x-h)-f(x)}{-h}\\ &= \lim_{(-h)\to 0^-}\frac{f(x)-f(x-h)}{h}.\end{aligned}$$

We may thus conclude that

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{(-h)\to 0}\frac{f(x)-f(x-h)}{h} = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h},$$

whence we have

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = \frac{1}{2}\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \frac{f(x)-f(x-h)}{h}\right) = \frac{1}{2}\cdot 2f'(x) = f'(x),$$

as was to be shown.

We now provide a counterexample. Setting $f(x) = |x|$, we see that any $|h| > 0$ yields

$$\frac{f(h)-f(-h)}{2h} = \frac{|h|-|-h|}{2h} = 0,$$

whence the above expression clearly converges to 0 as h approaches 0. f is not differentiable at 0, however. Indeed, the right limit of the differential is 1, and the left limit of the differential is -1, whence the limit does not exist. □

4.1.9. Suppose $f : (a, b) \to \mathbb{R}$ is continuous on (a, b) and differentiable at $x_0 \in (a, b)$. Define

$$g(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \in (a,b) \setminus \{x_0\};\\ f'(x_0) & \text{if } x = x_0.\end{cases}$$

Prove that g is continuous on (a, b).

Proof. Since f and $-f(x_0)$ are continuous on (a, b), and $1/(x - x_0)$ is continuous on $(a, b) \setminus \{x_0\}$, we see that g is continuous on $(a, b) \setminus \{x_0\}$. Moreover, f is differentiable at x_0, hence

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) = g(x_0),$$

whence g is continuous at x_0. It follows that g is continuous on (a, b), as was to be shown. □