SUGGESTED SOLUTIONS FOR PROBLEM SET 6

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3.1.1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x^2 - 2x + 1$. Show that f is continuous at 2.

Proof. Fixing $\varepsilon > 0$ and letting $\delta = \min\{1, \varepsilon/13\}$, we see that $|x - 2| < \delta$ implies

$$\begin{aligned} |3x^2 - 2x + 1 - (3 \cdot 2^2 - 2 \cdot 2 + 1)| &= |3x^2 - 2x - 8| \\ &= |(3x + 4)(x - 2)| \\ &= |3x + 4||x - 2| \\ &< |3x + 4|\delta \\ &\leq 13\delta \\ &\leq 13 \cdot \frac{\varepsilon}{13} \\ &= \varepsilon, \end{aligned}$$

as was to be shown. □

Remark. δ is either 1 or smaller, hence $|x - 2| < \delta$ stipulates that x is *definitely* between 1 and 3. The maximum value of $|3x + 4|$ on $[1, 3]$ is then $3 \cdot 3 + 4 = 13$, whence $|3x + 4|\delta \leq 13\delta$. Now, δ is either $\varepsilon/13$ or smaller, so we have $13\delta \leq 13 \cdot \varepsilon/13$.

3.1.8. Suppose $f : (a, b) \to \mathbb{R}$ is continuous and $f(r) = 0$ for each rational number $r \in (a, b)$. Prove that $f(x) = 0$ for all $x \in (a, b)$.

Proof. Suppose there exists an $x_0 \in (a, b)$ such that $f(x_0) > 0$. Setting $\varepsilon = f(x_0)/2$, we may find a δ such that $|x_0 - x| < \delta$ implies $|f(x_0) - f(x)| < \varepsilon$. Then

$$\frac{f(x_0)}{2} < f(x) < \frac{3f(x_0)}{2},$$

whence $f(x)$ is positive on $(x_0 - \delta, x_0 + \delta)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we may find a rational r in the interval $(x_0 - \delta, x_0 + \delta)$. This is absurd, for f was assumed to vanish at all rational points of (a, b). We may thus conclude that f vanishes at all points of (a, b).

Supposing $f(x_0) < 0$ produces a similar contradiction, hence we omit the proof. □

3.3.19. Let $f, g : D \to \mathbb{R}$ be uniformly continuous. Prove that $f + g : D \to \mathbb{R}$ is uniformly continuous. What can be said about fg? Justify.

Proof. Fix $\varepsilon > 0$, and find $\delta > 0$ such that $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |g(x) - g(y)| < \frac{\varepsilon}{2}$$

for all $x, y \in D$. Then $|x - y| < \delta$ implies

$$|(f + g)(x) - (f + g)(y)| = |f(x) + g(x) - f(y) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whence $f + g$ is uniformly continuous on D.

We claim that fg is uniformly continuous if f and g are bounded on D. We begin with a

Lemma. *Let $h : A \to \mathbb{R}$ be a uniformly continuous function. If h is bounded, the square h^2 is uniformly continuous.*

Proof of the lemma. Find $M > 0$ such that $|h(x)| < M$ for all $x \in A$. Fix $\varepsilon > 0$, and find $\delta > 0$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \varepsilon/2M$ for all $x, y \in A$. Then $|x - y| < \delta$ implies that

$$|h^2(x)^2 - h^2(y)| = |h(x) + h(y)||h(x) - h(y)| < 2M \cdot \frac{\varepsilon}{2M} = \varepsilon,$$

whence h^2 is uniformly continuous on A. □

By the lemma $(f+g)^2$ and $(f-g)^2$ are uniformly continuous if f and g are bounded. Therefore,

$$(f+g)^2 - (f-g)^2 = f^2 + 2fg + g^2 - f^2 + 2fg - g^2 = 4fg$$

is uniformly continuous, whence fg is uniformly continuous as well: tndeed, fixing $\varepsilon > 0$ and finding $\delta > 0$ such that $|x - y| < \delta$ implies $|4fg(x) - 4fg(y)| < \varepsilon/4$ for all $x, y \in D$, we see that $|x - y| < \delta$ also implies

$$|4fg(x) - 4fg(y)| = 4|fg(x) - fg(y)| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon,$$

as was claimed.

If the boundedness condition is dropped, the product fg may not be uniformly continuous even when both f and g are. For an example, we consider $f(x) = g(x) = x$ on $\mathbb{R}$. Fixing $\delta > 0$ and setting $\delta = \varepsilon$, we see that $|x - y| < \delta$ implies

$$|f(x) - f(y)| = |g(x) - g(y)| = |x - y| < \delta = \varepsilon$$

for all $x, y \in \mathbb{R}$. Set $h(x) = fg(x) = x^2$, and fix any $\varepsilon > 0$. Since $|x + y|$ may be made as large as desired, the statement

$$|h(x) - h(y)| = |x^2 - y^2| = |x + y||x - y| < |x + y|\delta < \varepsilon$$

does not hold for all $x, y \in \mathbb{R}$, no matter what δ is. It follows that $h = fg$ is not uniformly continuous. □

3.3.23. A function $f : \mathbb{R} \to \mathbb{R}$ is periodic iff there is a real number $h \neq 0$ such that $f(x+h) = f(x)$ for all $x \in \mathbb{R}$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is periodic and continuous, then f is uniformly continuous.

Proof. We first show that $[0, h]$ is compact. Given any $a \in \mathbb{R} \smallsetminus [0, h]$, we may set $\delta_a = \min\{a/2, (h - a)/2\}$ and observe that

$$(a - \delta_a, a + \delta_a) \subseteq \mathbb{R} \smallsetminus [0, h].$$

Hence, $\mathbb{R} \smallsetminus [0, h]$ is open, whence $[0, h]$ is closed. Since $[0, h]$ is clearly bounded, it follows from the Heine-Borel theorem that $[0, h]$ is compact.

Now, f is continuous on a compact set $[0, h]$, hence uniformly continuous on $[0, h]$. Fix $\varepsilon > 0$, and find $\delta_0 > 0$ such that $|x - y| < \delta_0$ implies $|f(x) - f(y)| < \varepsilon/2$ for all $x, y \in [0, h]$. We set $\delta = \min\{h/2, \delta_0\}$. Given any $x \in \mathbb{R}$, we may find an integer n_x and a real number $x_0 \in [0, h]$ such that

$$x = x_0 + n_x h.$$

Fix any $y \in \mathbb{R}$ such that $|x - y| < \delta$, and find an integer n_y and a real number $y_0 \in [0, h]$ such that

$$y = y_0 + n_y h.$$

If $n_x = n_y$, then $|x_0 - y_0| = |x - y| < \delta$, whence $|f(x) - f(y)| = |f(x_0) - f(y_0)| < \varepsilon/2 < \varepsilon$.

If $n_x = n_y$, then either $n_x = n_y + 1$ or $n_x + 1 = n_y$; indeed, $|x - y| < \delta \leq h/2$. Note that $|x_0 - 0| < \delta$ or $|x_0 - h| < \delta$, for otherwise $n_x = n_y$. We assume without loss of generality that $n_y = n_x + 1$, and that $|x_0 - h| < \delta$. The assumptions imply that $h \leq y - n_x h \leq 2h$ and that $|f(x_0) - f(h)| < \varepsilon/2$, respectively. Since $|x_0 - (y - n_x h)| = |(x - n_x h) - (y - n_x h)| = |x - y| < \delta$, we have $|h - (y - n_x h)| < \delta$. Therefore, $|f(h) - f(y - n_x h)| < \varepsilon/2$. It follows that

$$|f(x) - f(y)| = |f(x_0) - f(y - n_x h)| \leq |f(x_0) - f(h)| + |f(h) - f(y - n_x h)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as was to be shown. The other cases are analogous. □

3.3.26. Let $E \subseteq \mathbb{R}$. Prove that E is closed if, for every x_0 such that there is a sequence $(x_n)_{n=1}^{\infty}$ of points of E converging to x_0, it is true that $x_0 \in E$. In other words, prove E is closed if it contains all limits of sequences of members of E.

Proof. Suppose that x_0 is an accumulation point of E. For every $n \in \mathbb{N}$, we pick a point

$$x_n \in (x_0 - 1/n, x_0 + 1/n) \cap E.$$

This is a sequence in E converging to x_0, whence $x_0 \in E$ by the hypothesis. It follows that E is closed. □

3.3.27. Prove that every set of the form $\{x : a < x < b\}$ is open and every set of the form $\{x : a \leq x \leq b\}$ is closed.

Proof. For any $x_0 \in (a, b)$, we may set $\delta_{x_0} = \min\{(x_0 - a)/2, (b - x_0)/2\}$ and observe that

$$(x_0 - \delta_{x_0}, x_0 + \delta_{x_0}) \subseteq (a, b).$$

Hence, (a, b) is open. Given any $y_0 \in \mathbb{R} \smallsetminus [a, b]$, we may set $\delta_{y_0} = \min\{(y_0 - a)/2, (b - y_0)/2\}$ and observe that

$$(y_0 - \delta_{y_0}, y_0 + \delta_{y_0}) \subseteq \mathbb{R} \smallsetminus [a, b].$$

Hence, $\mathbb{R} \smallsetminus [a, b]$ is open, whence $[a, b]$ is closed. □

3.3.28. Let $E \subseteq \mathbb{R}$, and let D' be the set of accumulation points of D. Prove that $\bar{D} = D \cup D'$ is closed and that if F is any closed set that contains D, then $\bar{D} \subseteq F$. $\bar{D}$ is called the *closure* of D.

Proof. Suppose that x_0 is an accumulation point of $\bar{D}$, and let N' be any neighborhood of x_0. Then N' contains a point in D or a point in D'. If N contains a point in D', then N is a neighborhood of an accumulation point of D, whence it contains a point in D. Therefore, x_0 is an accumulation point of D, which implies that $x_0 \in D' \subseteq \bar{D}$. We conclude that $\bar{D}$ is closed.

Let F be any closed set containing D. If y_0 is an accumulation point of D, then any neighborhood of y_0 contains a point in D, hence in D. It follows that $D' \subseteq F$, whence $\bar{D} \subseteq F$. □

3.3.31. Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both continuous. Let $T = \{x : f(x) = g(x)\}$. Prove that T is closed.

A proof. We set $h = f - g$, so that

$$\{x : f(x) = g(x)\} = \{x : h(x) = 0\}.$$

Let $E = \{h(x) = 0\}$, and let a be an accumulation point of E. Take a sequence $(a_n)_{n=1}^{\infty}$ in E converging to a. h is continuous, hence $(h(a_n))_{n=1}^{\infty}$ converges to $h(a)$. Since $h(a_n) = 0$ for all $n \in \mathbb{N}$, we have $h(a) = 0$. It follows that $a \in E$, whence E is closed. □

Another proof: a topology magic. For any function $h : \mathbb{R} \to \mathbb{R}$ and a set $E \subseteq \mathbb{R}$, we define

$$h^{-1}(E) = \{x : f(x) \in E\}.$$

We establish a

Lemma. *If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $h^{-1}(E)$ is open in $\mathbb{R}$ for every open set $E \subseteq \mathbb{R}$.*

Proof of the lemma. Fix an open set E, and pick $x \in h^{-1}(E)$. Then $h(x) \in E$, so we may find $\varepsilon_x > 0$ such that $(h(x) - \varepsilon_x, h(x) + \varepsilon_x) \subseteq E$. Let $\delta_x > 0$ such that $|x - y| < \delta_x$ implies $|h(x) - h(y)| < \varepsilon_x$. This implies that $(x - \delta_x, x + \delta_x) \subseteq h^{-1}(E)$. Since x was arbitrary, $h^{-1}(E)$ is open. □

An immediate consequence is the following

Corollary. *If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $h^{-1}(K)$ is closed in $\mathbb{R}$ for every closed set $K \subseteq \mathbb{R}$.*

Proof of the corollary. K is closed, hence $\mathbb{R} \setminus K$ is open. The above Lemma then implies that $h^{-1}(\mathbb{R} \setminus K)$ is open. Since[1] $h^{-1}(\mathbb{R} \setminus K) = h^{-1}(\mathbb{R}) \setminus h^{-1}(K)$ and $h^{-1}(\mathbb{R}) = \mathbb{R}$, we see that $h^{-1}(K)$ is closed. □

We set $h = f - g$, so that

$$\{x : f(x) = g(x)\} = \{x : h(x) = 0\}.$$

We observe that $\{0\}$ is closed. In fact:

Lemma. *Every finite subset of $\mathbb{R}$ is compact.*

Proof of the lemma. Let $A = \{a_1, \ldots, a_n\}$ be a finite subset of $\mathbb{R}$, and let $\{U_\alpha\}$ be any open cover of A. For each $1 \leq i \leq n$, pick U_i in the collection $\{U_\alpha\}$ such that $a_i \in U_i$. Then $\{U_1, \ldots, U_n\}$ is clearly a finite subcover. □

Corollary. *Every finite subset of $\mathbb{R}$ is closed.*

Proof of the corollary. This follows from the above lemma and the Heine-Borel theorem. □

It now follows that

$$h^{-1}(\{0\}) = \{x : h(x) = 0\} = \{f(x) = g(x)\}$$

is closed, as was to be shown. □

Remark. Read the above "topology magic" carefully, especially the open-set characterization of continuous functions. Absorbing it will make your life much easier in dealing with continuous functions.

3.3.33. Find an open cover of $\{x : x > 0\}$ with no finite subcover.

Sketch of a proof. Let $A = \{(0, n) : n \in \mathbb{N}\}$. If we fix n, then $n + 1$ is not covered. □

3.3.34. Find an open cover of $(1, 2)$ with no finite subcover.

Sketch of a proof. Let $A = \{(1 + 1/(n + 1), 1 + 1/n) : n \in \mathbb{N}\}$. □

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[1] $x \in h^{-1}(\mathbb{R} \setminus K)$ iff $h(x) \in \mathbb{R} \setminus K$ iff $h(x) \in \mathbb{R}$ and $h(x) \notin K$ iff $x \in h^{-1}(\mathbb{R})$ and $x \notin h^{-1}(K)$ iff $x \in h^{-1}(\mathbb{R}) \setminus h^{-1}(K)$.