SUGGESTED SOLUTIONS FOR PROBLEM SET 5

FALL 2010, MATH 311:01

2.1.8. Define $f : (0,1) \to \mathbb{R}$ by $f(x) = \dfrac{x^3 - x^2 + x - 1}{x - 1}$. Prove that f has a limit at 1.

Proof. We observe that

$$f(x) = \frac{x^3 - x^2 + x - 1}{x - 1} = \frac{x^2(x-1) + (x-1)}{x-1} = x^2 + 1$$

for all $x \in (0,1)$. We claim that $f(x)$ converges to 2 as x approaches 1. First, we note that $|x+1| \leq 2$ for all $x \in (0,1)$. Fix $\varepsilon > 0$, and set $\delta = \varepsilon/2$. We then see that $|x-1| < \delta$ implies

$$|(x^2+1) - 2| = |x^2 - 1| = |x+1||x-1| \leq 2|x-1| < 2\delta = \varepsilon,$$

as was to be shown. □

2.2.12. Suppose $f : D \to \mathbb{R}$ has a limit at x_0. Prove that $|f| : D \to R$ has a limit at x_0 and that $\lim_{x \to x_0} |f(x)| = \lim_{x \to x_0} |f(x)|$.

A sequence proof. Let $L = \lim_{x \to x_0} f(x)$. Then, for every sequence $(x_n)_{n=1}^\infty$ converging to x_0, the sequence $(f(x_n))_{n=1}^\infty$ converges to L. Fix an arbitrary sequence $(x_n)_{n=1}^\infty$. Since $|x| = \sqrt{x^2}$ for any real number x, the sequence $(|f(x_n)|)_{n=1}^\infty$ is precisely $\left(\sqrt{f(x_n)^2}\right)_{n=1}^\infty$, which converges to $\sqrt{L^2} = |L|$; here we have used properties of the arithmetic operations on sequences. $(x_n)_{n=1}^\infty$ was arbitrary, thus we may conclude that $|f(x)|$ converges to $|L|$ as x approaches x_0. □

A triangle-inequality proof. Let $L = \lim_{x \to x_0} f(x)$. Fix $\varepsilon > 0$, and find $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$. We recall that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$, and observe the following:

$$||f(x)| - |L|| \leq |f(x) - L| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we may conclude that

$$\lim_{x \to x_0} |f(x)| = |L| = \left| \lim_{x \to x_0} f(x) \right|,$$

as was to be shown. □

2.2.13. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x - [x]$. Determine these points at which f has a limit, and justify your conclusions.

Sketch of a proof. We claim that f has a limit at x_0 whenever x_0 is not an integer. Verify that $f(x)$ converges to $f(x_0)$ as x approaches x_0 when x_0 is not an integer by setting

$$\delta = \min\{\varepsilon, |[x_0 + 1] - x_0|.|x_0 - [x_0]|\}.$$

Proceed as in Exercise 2.1.3 to show that f does not have a limit at x_0 when x_0 is an integer; f has a jump of length 1 at x_0, so we may set $\varepsilon = 1/2$. (*Hint*: Draw the graph!) □

2.3.18. Define $g : (0,1) \to \mathbb{R}$ by $g(x) = \dfrac{\sqrt{1+x}-1}{x}$. Prove that g has a limit at 0 and find it.

Proof. We observe that

$$g(x) = \frac{\sqrt{1+x}-1}{x} = \frac{\sqrt{1+x}-1}{(1+x)-1} = \frac{\sqrt{1+x}-1}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} = \frac{1}{\sqrt{1+x}+1} \tag{1}$$

for all $x \in (0,1)$. We claim that $g(x)$ converges to $1/2$ as x approaches zero. First, we note that $|\sqrt{1+x}+1| > 2$ for all $x \in (0,1)$, so that

$$\left|\frac{1}{\sqrt{1+x}+1}\right| < \frac{1}{2}.$$

Now, fix $\varepsilon > 0$, and set $\delta = (\varepsilon+1)^2 - 1$. Then, for any $x > 0$, we see that $x < \delta$ implies $\sqrt{1+x} < \sqrt{1+\delta}$, so that

$$\sqrt{1+x}-1 < \sqrt{1+\delta}-1 = \sqrt{1+(\varepsilon+1)^2-1}-1 = \varepsilon+1-1 = \varepsilon.$$

In particular, $|\sqrt{1+x}-1| > 0$ for all $x > 0$, hence $|\sqrt{1+x}-1| < \varepsilon$. Therefore,

$$\begin{aligned}
\left|\frac{1}{2} - \frac{1}{\sqrt{1+x}+1}\right| &= \left|\frac{\left(\sqrt{1+x}+1\right)-2}{2\left(\sqrt{1+x}+1\right)}\right| \\
&= \frac{\left|\left(\sqrt{1+x}+1\right)-2\right|}{|2|\left|\left(\sqrt{1+x}+1\right)\right|} \\
&< \left|\left(\sqrt{1+x}+1\right)-2\right| \\
&= \left|\sqrt{1+x}-1\right| \\
&< \varepsilon,
\end{aligned}$$

as was to be shown. □

Remark. Alternatively, you could make the reduction in (1), and compute the limit of the numerator and of the denominator in view of Theorem 2.4. In this case, there must be an appropriate justification that $\sqrt{1+x}+1$ indeed converges to 2 as x approaches 0.

2.20. Prove Theorem 2.5:

Theorem. *Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, x_0 is an accumulation point of D, and f and g have limits at x_0. If $f(x) \le g(x)$ for all $x \in D$, then*

$$\lim_{x\to x_0} f(x) \le \lim_{x \to x_0} g(x).$$

Proof. We set

$$L_1 = \lim_{x\to x_0} f(x) \qquad \text{and} \qquad L_2 = \lim_{x\to x_0} g(x).$$

Pick any sequence $(x_n)_{n=1}^{\infty}$ converging to x, and observe that $(f(x_n))_{n=1}^{\infty}$ converges to L_1, and $(g(x_n))_{n=1}^{\infty}$ to L_2. Since $f(x_n) \le g(x_n)$ for all $n \in \mathbb{N}$, we have $L_1 \le L_2$ by Theorem 1.12. This completes the proof. □