SUGGESTED SOLUTIONS FOR PROBLEM SET 4

FALL 2010, MATH 311:01

1.4.35. Suppose x is an accumulation point of $\{a_n : a \in J\}$. Show that there is a subsequence of $(a_n)_{n=1}^{\infty}$ that converges to x.

Proof. We inductively define a subsequence $(a_{n_k})_{k=1}^{\infty}$ as follows. First, we let $a_{n_1} = a_1$. Having defined $a_{n_1}, \ldots, a_{n_{k-1}}$, we set

$$d_k = (1/2) \cdot \min\{|x - a_i| : 1 \leq i \leq n_{k-1}\}$$

and define a_{n_k} to be any point in the intersection $(x - d_k, x + d_k) \cap \{a_n : a \in J\}$; the sequence is well-defined, since x is an accumulation point of $\{a_n : a \in J\}$.

We claim that $(a_{n_k})_{k=1}^{\infty}$ converges to x. We first note that any $\varepsilon > 0$ admits at least one point $a_n \in (x - \varepsilon, x + \varepsilon) \cap \{a_n : a \in J\}$. Find any k such that $n_k > n$. By construction, we have

$$\varepsilon \geq |x - a_n| > |x - a_{n_k}|,$$

so that $a_{n_k} \in (x - \varepsilon, x + \varepsilon)$. Furthermore, we have $|a_{n_k} - x| > |a_{n_l} - x|$ for any $l > k$, whence $a_{n_l} \in (x - \varepsilon, x + \varepsilon)$ for all $l > k$. It follows that the subsequence convergest to x, as was claimed. □

1.4.36. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Prove that $(a_n)_{n=1}^{\infty}$ has a convergent subsequence. (*Hint*: You may want to use the Bolzano-Weierstrass Theorem)

Proof. By Bolzano-Weierstrass Theorem, the set $\{a_n : a \in \mathbb{N}\}$ has an accumulation point x. By Exercise 1.4.35, there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ converging to x. □

Remark. The Bolzano-Weierstrass theorem is a characterization of *sequential compactness* in Euclidean spaces. A set $X \subseteq \mathbb{R}$ is *sequentially compact* if every sequence contained in X has a subsequence converging to a point in X. The Bolzano-Weierstrass theorem, in this terminology, states that a set $X \subseteq \mathbb{R}$ is sequentially compact if and only if X is closed—i.e., X contains all of its accumulation points—and bounded. Defining the *closure* of a set $X \subseteq \mathbb{R}$ to be the union $X \cup X'$, where X' is the collection of all accumulation points of X', we may observe further that the assertion in Exercise 1.4.36 is "the closure of $\{a_n : n \in \mathbb{N}\}$ is sequentially compact, provided that $(a_n)_{n=1}^{\infty}$ is bounded."

1.4.38. Prove that if $c > 1$, then $(\sqrt[n]{c})_{n=1}^{\infty}$ converges to 1.

Proof. For each $n \in \mathbb{N}$, we have $\sqrt[n]{c} > 1$, and the sequence is bounded below. Furthermore, $c > 1$ implies that

$$\sqrt[n]{c} - \sqrt[n-1]{c} = \sqrt[n]{c}(1 - \sqrt[n(n-1)]{c}) < 0,$$

so that the sequence is monotone-decreasing. $(\sqrt[n]{c})_{n=1}^{\infty}$ is therefore a convergent sequence. We call the limit L. We shall now establish a preliminary

Lemma. *If $(a_n)_{n=1}^{\infty}$ converges to a with $a_n \geq 0$ for all n, show $(\sqrt{a_n})_{n=1}^{\infty}$ converges to $\sqrt{a}$.*

Proof of the lemma. We note that

$$\sqrt{a_n} - \sqrt{a} = \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}}$$

for each $n \in \mathbb{N}$. $(a_n)_{n=1}^\infty$ is a convergent sequence, hence it is bounded, and we may set

$$m = \inf\{a_n : n \in \mathbb{N}\}.$$

Fix $\varepsilon > 0$, and find N such that $n > N$ implies $|a_n - a| < (\sqrt{m} + \sqrt{a})\varepsilon$. Then we have, for each $n > N$,

$$|\sqrt{a_n} - \sqrt{a}| = \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| \leq \left| \frac{a_n - a}{\sqrt{m} + \sqrt{a}} \right| < \frac{1}{\sqrt{m} + \sqrt{a}} \cdot (\sqrt{m} + \sqrt{a})\varepsilon = \varepsilon,$$

thus establishing our lemma. □

By the lemma, $(\sqrt{\sqrt[n]{c}})_{n=1}^\infty$ converges to $\sqrt{L}$. Now, $(\sqrt{\sqrt[n]{c}})_{n=1}^\infty = (\sqrt[2n]{c})_{n=1}^\infty$ is a subsequence of a convergent sequence $(\sqrt[n]{c})_{n=1}^\infty$, whence it converges to L. The limit of a sequence is unique, hence $L = \sqrt{L}$. Since $\sqrt[n]{c} > 1$ for all $n \in \mathbb{N}$, L cannot be 0. We may thus conclude that $L = 1$. □

1.4.45. Show that if x is any real number, there is a sequence of rational numbers converging to x.

Proof. We inductively define a sequence $(x_n)_{n=1}^\infty$ as follows. Let $x_1 = 2^{43112609} - 1$. Having defined $x_1, \ldots, x_{n-1}$, we define x_n to be any rational number between x and $x + 1/n$. The sequence is well-defined, as there is a rational number between any two distinct real numbers.

We show that $(x_n)_{n=1}^\infty$ converges to x. Indeed, given any $\varepsilon > 0$, we can find an integer N such that $1/N < \varepsilon$. Hence, $a_n \in (x - \varepsilon, x + \varepsilon)$ for all $n > N$, thereby establishing the convergence. □

1.4.47. Suppose that $(a_n)_{n=1}^\infty$ converges to A and that B is an accumulation point of $\{a_n : n \in J\}$.

Proof. As per Exercise 1.4.35, we may extract a subsequence $(a_{n_k})_{k=1}^\infty$ converging to B. Since $(a_n)_{n=1}^\infty$ is a convergent sequence, every subsequence of $(a_n)_{n=1}^\infty$ must converge to the same limit, whence $A = B$. □

2.1.1. Define $f : (-2, 0) \to \mathbb{R}$ by $f(x) = \dfrac{x^2 - 4}{x + 2}$. Prove that f has a limit at -2, and find it.

Proof. We observe that

$$f(x) = \frac{(x-2)(x+2)}{x+2} = x - 2$$

for all $x \in (-2, 0)$. We claim that $f(x)$ converges to -4 as x approaches -2. Indeed, given any $\varepsilon > 0$, we may set $\delta = \varepsilon$ to see that $|(-2) - x| < \delta$ implies

$$|(-4) - f(x)| = |(-4) - (x - 2)| = |(-2) - x| < \delta = \varepsilon,$$

as desired. □

Remark. This confirms the calculus intuition that the value at a point does not matter for computing the limit at the point. For example, defining $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} c & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we see that

$$\lim_{x \to 0} f(x) = 0,$$

regardless of what c may be.

2.1.2. Define $f : (-2, 0) \to \mathbb{R}$ by $f(x) = \dfrac{2x^2 + 3x - 2}{x + 2}$. Prove that f has a limit at -2, and find it.

Proof. We observe that
$$f(x) = \frac{(x + 2)(2x - 1)}{x + 2} = 2x - 1$$
for all $x \in (-2, 0)$. Proceed as in Exercise 2.1.1. $\square$

2.1.3. Give an example of a function $f : (0, 1) \to \mathbb{R}$ that has a limit at every point of $(0, 1)$ except $\frac{1}{2}$. Use the defintion of *limit of a function* to justify the example.

Proof. We set
$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$
If $x < 1/2$, then any $\varepsilon > 0$ furnishes $\delta < |1/2 - x|$ such that $|x - y| < \delta$ implies
$$|f(x) - 0| = |0 - 0| = 0 < \varepsilon,$$
whence f converges to 0 at x. Similarly, if $x > 1/2$, then any $\varepsilon > 0$ furnishes $\delta < |1/2 - x|$ such that $|x - y| < \delta$ implies
$$|f(x) - 1| = |1 - 1| = 0 < \varepsilon,$$
whence f converges to 1 at x.

Set $x = 1/2$, fix $\varepsilon = 1/2$ and let L be any real number. If $L \notin (-1/2, 3/2)$, then $|f(x) - L| \ge \varepsilon$ for any x, so that f does not converge to L at x. If $L \in [1/2, 3/2)$, then $x < 1/2$ implies that
$$|f(x) - L| = |0 - L| \ge 1/2 = \varepsilon,$$
whence f does not converge to L at x. Finally, if $L \in (-1/2, 1/2)$, then $x > 1/2$ implies that
$$|f(x) - L| = |1 - L| \ge 1/2 = \varepsilon,$$
whence f does not converge to L at x. It follows that f does not have a limit at $x = 1/2$. $\square$