SUGGESTED SOLUTIONS FOR PROBLEM SET 3

FALL 2010, MATH 311:01

1.1.1. Show that $[0,1]$ is a neighborhood of $\frac{2}{3}$—that is, there is $\varepsilon > 0$ such that

$$\left(\frac{2}{3} - \varepsilon, \frac{2}{3} + \varepsilon\right) \subseteq [0,1].$$

Proof. We set $\varepsilon = 1/3$, and observe that

$$\left(\frac{2}{3} - \varepsilon, \frac{2}{3} + \varepsilon\right) = \left(\frac{1}{3}, 1\right) \subseteq [0,1].$$

□

1.1.5. Give an example of a sequence that is bounded but not convergent.

Proof. For each $n \in \mathbb{N}$, we set $a_n = (-1)^n$. Then, the sequence $(a_n)_{n=1}^{\infty}$ is entirely contained in $[-1,1]$, hence bounded. Setting $\varepsilon = 1/2$, however, there is no real number A such that $|a_n - A| < \varepsilon$ for all sufficiently large n. To see this, we note that 1 and -1 cannot both be in $(A - 1/2, A + 1/2)$ for any $A \in \mathbb{R}$. Then, for any N, there exists an $n > N$ such that $|a_n - A| \geq \varepsilon$: if -1 is in $(A - 1/2, A + 1/2)$, we pick $n = 2N$; if 1 is in $(A - 1/2, A + 1/2)$, we pick $n = 2N + 1$. This proves that $(a_n)_{n=1}^{\infty}$ does not converge. □

1.1.6b. Use the definition of convergence to prove that each of the following sequences converges:

$$\left(\frac{2 - 2n}{n}\right)_{n=1}^{\infty}.$$

Proof. We claim that the above sequence converges to -2. We first note that

$$\left|\frac{2 - 2n}{n} - (-2)\right| = \left|\frac{2 - 2n + 2n}{n}\right| = \left|\frac{2}{n}\right|$$

for any $n \in \mathbb{N}$. Fix $\varepsilon > 0$, and invoke the archimedean principle to pick an integer $N > 2/\varepsilon$. We now observe that any $n > N$ yields

$$\left|\frac{2 - 2n}{n} - (-2)\right| = \left|\frac{2}{n}\right| < \left|\frac{2}{N}\right| < \varepsilon,$$

whence the above sequence converges to -2 as claimed. □

1.2.17. Prove that the sequence $\left(\frac{2n+1}{n}\right)_{n=1}^{\infty}$ is Cauchy.

Proof. We claim that the above sequence converges to 2. We first note that

$$\left|\frac{2n+1}{n} - 2\right| = \left|\frac{2n + 1 - 2n}{n}\right| = \left|\frac{1}{n}\right|$$

for any $n \in \mathbb{N}$. Fix $\varepsilon > 0$, and invoke the archimedean principle to pick an integer $N > 1/\varepsilon$. We now observe that any $n > N$ yields

$$\left|\frac{2n+1}{n} - 2\right| = \left|\frac{1}{n}\right| < \left|\frac{1}{N}\right| < \varepsilon,$$

whence the above sequence converges to as claimed.

Recalling that every convergent sequence in $\mathbb{R}$ is Cauchy, we conclude that the above sequence is Cauchy. □

Remark. The equivalence of Cauchy sequences and convergent sequences is a manifestation of the *completeness* of the field $\mathbb{R}$ of real numbers. In mathematical analysis, we typically construct $\mathbb{R}$ from the field $\mathbb{Q}$ of rational numbers via *Dedekind cuts* or *Cauchy completion*: the Dedekind-cut approach fills in the supremums of bounded-above sets, whereas the Cauchy-completion approach fills in the limits of Cauchy sequences.

1.2.18. Give an example of a set with exactly two accumulation points.

Proof. For each $n \in \mathbb{N}$, we define $K_n = \{n + 1/m : m \in \mathbb{N}\}$. For example,

$$K_5 = \left\{5+1, 5+\frac{1}{2}, 5+\frac{1}{3}, 5+\frac{1}{4}, \cdots\right\}.$$

We claim that each K_n has precisely one accumulation point, namely n, and

$$\bigcup_{n=1}^{N} K_n$$

has N accumulation points: $1, 2, \ldots, N$. An immediate consequence is that

$$\bigcup_{n=1}^{\infty} K_n$$

has countably many accumulation points.

Fix $n \in \mathbb{N}$. For any $\varepsilon > 0$, we may find an integer $M > 1/\varepsilon$, so that every $m > M$ implies $1/m < \varepsilon$. This, in particular, implies that $n + 1/m$ is in $(n-\varepsilon, n+\varepsilon)$. Since ε was arbitrary, every neighborhood of n contains infinitely many points of K_n; in fact, it contains all but finitely many points of K_n. It follows that n is an accumulation point of K_n. If n' is any real number distinct from n, then we may pick $0 < \varepsilon_0 < |n - n'|/2$, and examine the open interval $(n' - \varepsilon_0, n' + \varepsilon_0)$. Since the open interval $(n - \varepsilon_0, n + \varepsilon_0)$, disjoint from $(n' - \varepsilon_0, n' + \varepsilon_0)$, contains all but finitely many points of K_n, the open interval $(n' - \varepsilon_0, n' + \varepsilon_0)$ about n' cannot contain infinitely many points of K_n. It follows that n' is not an accumulation point of K_n, whence we conclude that n is the only accumulation point of K_n.

By similar reasoning, we can show that the accumulation points of

$$\bigcup_{n=1}^{N} K_n$$

are $1, 2, \ldots, N$; in fact, reproducing the above proof N times would suffice. It follows, in particular, that $K_1 \cup K_2$ is an example of a set with exactly two accumulation points. □

Remark. *Infinitely many* and *all but finitely many* are different—this is an important distinction to make. Any neighborhood of the limit of a sequence contains all but finitely many terms; any neighborhood of an accumulation point of a set contains infinitely many points.

1.2.22. Let S be a nonempty set of real numbers that is bounded from above (below) and let $x = \sup S$ ($\inf S$). Prove that either x belongs to S or x is an accumulation point of S.

Proof. We argue by contraposition. Suppose that x is neither a point of x nor an accumulation point of a bounded-above set S. We show that x cannot be a supremum of S. Indeed, we may find an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ does not intersect S, whence x does not belong to S. We may suppose that x is an upper bound of S without loss of generality, since no real number can be a supremum of a set without being an upper bound of the set.

We now show that $x - \varepsilon$ is an upper bound of S, whence x cannot be a supremum of S. We first note that each $s \in S$ satisfies $s \leq x$. Since $(x - \varepsilon, x + \varepsilon)$ does not intersect S, we are forced to conclude that $x \leq x - \varepsilon$. Therefore, $x - \varepsilon$ is an upper bound, as desired.

We have thus shown that x is neither a point of x nor an accumulation point of S, from which the desired result follows. The case for $x = \inf S$ is similar. □

Remark. Therefore, x belongs to the *closure* of S, which is the collection of the points in S and the accumulation points of S. As we shall see later, the closure of any subset of $\mathbb{R}$ is *closed*.

1.3.25. Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences such that $(a_n)_{n=1}^{\infty}$ and $(a_n + b_n)_{n=1}^{\infty}$ converge. Prove that $(b_n)_{n=1}^{\infty}$ converges.

Proof. Let A be the limit of $(a_n)_{n=1}^{\infty}$, and C the limit of $(a_n + b_n)_{n=1}^{\infty}$. Fix $\varepsilon > 0$, and find integers N_1 and N_2 such that $n_1 > N_1$ and $n_2 > N_2$ imply

$$|a_{n_1} - A| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |(a_{n_2} + b_{n_2}) - C| < \frac{\varepsilon}{2},$$

respectively. Set $N = \max\{N_1, N_2\}$, so that each $n > N$ implies

$$|a_n - A| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |(a_n + b_n) - C| < \frac{\varepsilon}{2}.$$

Then, an application of the triangle inequality yields

$$|b_n - (A - C)| = |((a_n + b_n) - C) - (a_n - A)| \leq |(a_n + b_n) - C| + |a_n - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n > N$, whence $(b_n)_{n=1}^{\infty}$ converges to $A - C$. □

1.3.26. Give an example in which $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ do not converge but $(a_n + b_n)_{n=1}^{\infty}$ converges.

Proof. For each $n \in \mathbb{N}$, we set $a_n = (-1)^n$, and $b_n = (-1)(-1)^n = (-1)^{n+1}$. We have shown in Exercise 1.1.5 that neither $(a_n)_{n=1}^{\infty}$ nor $(b_n)_{n=1}^{\infty}$ converges. Now, $a_n + b_n = 0$ for all $n \in \mathbb{N}$, whence $(a_n + b_n)_{n=1}^{\infty}$ clearly converges to 0. □

1.3.32b. Find the limit of the sequence with the general term as given:

$$\frac{\cos n}{n}$$

Proof. We have proved in Exercise 1.2.18 that the sequence $(1/n)_{n=1}^{\infty}$ converges to 0. We know that $|\cos n| \leq 1$ for all $n \in \mathbb{N}$, whence it follows from Theorem 1.13 that the sequence converges to 0. □