SUGGESTED SOLUTIONS FOR PROBLEM SET 2

FALL 2010, MATH 311:01

0.5.43. Let $A = \{r : r \text{ is a rational number and } r^2 < 2\}$. Prove that A has no largest member. (*Hint*: If $r^2 < 2$, and $r > 0$, choose a rational number δ such that $0 < \delta < 1$ and $\delta < \dfrac{2 - r^2}{2r+1}$. Show that $(r+\delta)^2 < 2$.)

Proof. We suppose for a contradiction that A has a largest member, namely r. Since $r > 0$ and $r^2 < 2$, we have $2 - r^2 > 0$ and $2r + 1 > 0$, so that we may find a rational number δ such that $0 < \delta < 1$ and $\delta < (2 - r^2)/(2r + 1)$. A rearrangement of the second inequality yields $r^2 + \delta(2r + 1) < 2$, or $r^2 + 2r\delta + \delta < 2$. The first inequality implies that $\delta^2 < \delta$, whence

$$(r + \delta)^2 = r^2 + 2r\delta + \delta^2 < r^2 + 2r\delta + \delta < 2.$$

It follows that $r + \delta \in A$; this is absurd, as r was assumed to be the largest member of A. We conclude that A dose not have a largest member. □

Remark. This shows that the field $\mathbb{Q}$ of rational numbers is not complete. The field $\mathbb{R}$ of real numbers is the *completion* of $\mathbb{Q}$, in the sense that every bounded-above subset of $\mathbb{R}$ has a least upper bound.

0.5.44. If $x = \sup S$, show that, for each $\varepsilon > 0$, there is $a \in S$ such that $x - \varepsilon < a \leq x$.

A direct proof. Fix any $\varepsilon > 0$, so that $x - \varepsilon < x$. Since x is the least upper bound of S, it follows that $x - \varepsilon$ is not an upper bound, viz., there exists an element $a \in S$ such that $x - \varepsilon \leq a$. Noting that the least upper bound of a set is no smaller than any of the elements in the set, we may conclude that the a we have furnished satisfies the inequality $x - \varepsilon \leq x < x$. □

A proof by contradiction. We suppose for a contradiction that $\varepsilon > 0$ does not satisfy $x - \varepsilon < a \leq x$ for any $a \in S$. Then $x - \varepsilon \geq a$ for all $a \in S$, whence $x - \varepsilon$ is an upper bound of S such that $x - \varepsilon < x$. This contradicts the definition of least upper bound, and it follows that every $\varepsilon > 0$ furnishes an $a \in S$ such that $x - \varepsilon < a \leq x$. □

Remark. This shows that the least upper bound of a subset of $\mathbb{R}$ is a *limit point* of the set, viz., every open interval containing the supremum contains an element of the set.

0.5.45. If $y = \inf S$, show that, for each $\varepsilon > 0$, there is $a \in S$ such that $y \leq a < y + \varepsilon$.

A direct proof. Fix any $\varepsilon > 0$, so that $y - \varepsilon > y$. Since y is the greatest lower bound of S, it follows that $y - \varepsilon$ is not a lower bound, viz., there exists an element $a \in S$ such that $y - \varepsilon \geq a$. Noting that the greatest lower bound of a set is no bigger than any of the elements in the set, we may conclude that the a we have furnished satisfies the inequality $y - \varepsilon \geq y > y$. □

A proof by contradiction. We suppose for a contradiction that $\varepsilon > 0$ does not satisfy $y + \varepsilon > a \geq y$ for any $a \in S$. Then $y + \varepsilon \leq a$ for all $a \in S$, whence $y + \varepsilon$ is a lower bound of S such that $y + \varepsilon > y$.

This contradicts the definition of greatest lower bound, and it follows that every $\varepsilon > 0$ furnishes an $a \in S$ such that $y + \varepsilon > a \geq y$. □

Remark. This shows that the greatest lower bound of a subset of $\mathbb{R}$ is a limit point of the set.

Exercise 1. Show that there are no negative numbers in the intersection of $(-1/n, 1/n)$ where n goes from 1 to infinity.

Proof. We first show that every negative real number x admits a natural number n such that $x < -1/n$. To show this, we invoke theorem 0.22 to find a rational number q between x and 0. By the definition of rational numbers, we may find positive integers p_1 and p_2 such that $q = -p_1/p_2$. We now observe that

$$x < -\frac{p_1}{p_2} \leq -\frac{1}{p_2} < 0,$$

whence the desired result follows.

Now, if x is any negative real number, then we may find a natural number n such that $x < -1/n$. Hence, $x \notin (-1/n, 1/n)$, and we see that

$$x \notin \bigcap_{n=1}^{\infty} (-1/n, 1/n).$$

Since x was arbitrary, the proof is complete. □

Remark. The preliminary result proved above is a special case of the following theorem:

Theorem (The Archimedean Property)**.** *If x is a positive real number, and y a real number, then there exists a natural number n such that $nx > y$.*

Exercise 2. Prove that $2n + 1 < 2^n$ for $n \geq 3$.

Proof. We proceed by mathematical induction on n. Clearly, $2 \cdot 3 + 1 < 2^3$. If we suppose that the statement holds for some fixed $n \geq 3$, then we see that

$$2(n+1) + 1 = (2n+1) + 2 < 2^n + 2 < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1};$$

note that we have used $2 < 2^n$, which is true for all $n > 1$. □

Exercise 3. Prove that there is only one greatest lower bound.

Proof. Let $S \subseteq \mathbb{R}$, and suppose that p and q are greatest lower bounds of S. p is a lower bound of S, hence $p \leq q$. Likewise, $q \leq p$, and it follows that $p = q$. □