SUGGESTED SOLUTIONS FOR PROBLEM SET 1

FALL 2010, MATH 311:01

0.5.39. If $x < y$, prove that $x < \dfrac{x+y}{2} < y$.

Proof. Observe that $x + x < x + y < y + y$. Dividing through by 2 yields the desired result. $\square$

0.5.40. If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \dfrac{x+y}{2}$. (Hint: Use the fact $(\sqrt{x} - \sqrt{y})^2 \geq 0$).

Proof. We know that $x + y - 2\sqrt{xy} = (\sqrt{x} - \sqrt{y})^2 \geq 0$. Rearrange the inequality to obtain $x + y \geq 2\sqrt{xy}$. Dividing through by 2 now yields the desired result. $\square$

0.5.41. If $0 < a < b$, prove that $0 < a^2 < b^2$ and $0 < \sqrt{a} < \sqrt{b}$.

Proof. We first note that $0 < a < b$ implies $0 < aa < ab < bb$, which establishes the first inequality. To show the second inequality, we suppose that $\sqrt{a} \geq \sqrt{b}$; we may assume further that $\sqrt{a} \geq \sqrt{b} > 0$, since the square root of a positive real number is a positive real number. It immediately follows that the inequality

$$0 < \sqrt{b}\sqrt{b} \leq \sqrt{b}\sqrt{a} \leq \sqrt{a}\sqrt{a}$$

holds, whence $b \leq a$. We conclude that $0 < a < b$ implies $0 < \sqrt{a} < \sqrt{b}$, as desired. $\square$

0.5.42. If x, y, a and b are greater than zero and $\dfrac{x}{y} < \dfrac{a}{b}$, prove that $\dfrac{x}{y} < \dfrac{x+a}{y+b} < \dfrac{a}{b}$.

The obvious cross-multiplication proof. Observe that $y > 0$ and $b > 0$ imply $bx < ay$. Thus, $bx + xy < xy + ay$, which in turn yields $x(b+y) < y(x+a)$. Since x, y, $x+a$, and $y+b$ are positive, we may conclude that

$$\frac{x}{y} < \frac{x+a}{y+b}.$$

Similarly, $bx + ab < ay + ab$, and we have $b(x + a) < a(y + b)$. Since a, b, $x + a$, and $y + b$ are positive, it follows that

$$\frac{x+a}{y+b} < \frac{a}{b}.$$

Combining the two inequalities yields the desired result. $\square$

Another proof, just for fun. We first prove the following

Lemma. *If x, y, a and b are greater than zero and $\dfrac{x}{y} = \dfrac{a}{b}$, then we have*

$$\frac{x}{y} = \frac{x+a}{y+b} = \frac{a}{b}.$$

Proof of the lemma. We find a positive real number k such that $a = kx$ and $b = ky$. Then

$$\frac{x+a}{y+b} = \frac{x+kx}{y+ky} = \frac{(k+1)x}{(k+1)y} = \frac{x}{y} = \frac{a}{b},$$

as desired. □

Find a positive real number c such that $\frac{x}{y} = \frac{c}{b}$. Similarly, find a positive real number d such that $\frac{a}{b} = \frac{d}{y}$. The inequality $\frac{c}{b} = \frac{x}{y} < \frac{a}{b} = \frac{d}{y}$ yields the two inequalities $c < a$ and $x < d$, whence it follows from the above lemma that

$$\frac{x}{y} = \frac{x+c}{y+b} < \frac{x+a}{y+b} < \frac{d+a}{y+b} = \frac{a}{b}.$$

We note that we have applied the lemma twice. □